The Interplay Between Professional Noticing and Knowledge: The Case of Whole Number Multiplication

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The aim of the study reported in this article was to reveal how instruction, enriched with reasoning about students' thought processes and strategies, regulates prospective teacher's noticing skills in the context of whole number multiplication. In addition, the study examined the evidence of the prospective teacher's (Matt) knowledge that supported the interplay between knowledge and noticing that emerged during the learning process. To this end, the prospective teacher participated in the instruction enriched with reasoning about students' thought processes and strategies. The prospective teacher's noticing skills related to correct and incorrect student-invented strategies were examined before and after his engagement with the instruction provided. The data gathered through the prospective teacher's written responses in the pre and post-tests and the follow up interviews were analysed with respect to the dimensions of the Professional Noticing of Children's Mathematical Thinking framework. Moreover, to explore the evidence of knowledge that underpins the teacher's noticing skills, data were analysed through the Mathematics Teacher's Specialised Knowledge model. The findings indicated positive change in the prospective teacher's noticing skills for both correct and incorrect student-invented strategies after his involvement in the intervention.

Keywords • teacher knowledge • teacher noticing • whole number multiplication • student-invented strategies • prospective teachers

Introduction

Effective teaching requires orchestration by teachers with high levels of teacher competence to produce high levels of student learning (Hino et al., 2017). Such teacher competence includes professional teacher knowledge, professional beliefs, motivation, self-regulation and noticing (Hino et al., 2017). The central point of teacher competence is regarded as teacher knowledge (Schwarz & Kaiser, 2016), which was based on the seminal work of Shulman (1986). The researchers agreed that teachers need to have deep knowledge to orchestrate effective mathematics teaching. Important issues that researchers and educators contemplate, however, are how and when the teachers use their knowledge during planning and conducting instruction (Lampert, 2001). Thus, recent researchers have begun to explore the practice of teaching, in which teachers identify students' thinking, then decide whether the unexpected events are noteworthy and require the teacher to address them or not, and instantly evaluate the progress of lesson (Sherin et al., 2008). Considering this perspective, the researchers suggested the construct of teacher noticing, which in general is defined as what a teacher sees and responds to as it occurs in a classroom (Jacobs & Spangler, 2017).

Teachers should be aware of noteworthy events that occur during their instruction and employ their knowledge to deal with such events. However, it is an extremely challenging issue for teachers to be aware of those events while implementing the instruction (Jacobs et al., 2010) because the classroom environment has a complex structure in which many events occur simultaneously (Sherin et al., 2011a). In this regard, mathematics educators (e.g., Jacobs et al., 2010; van Es & Sherin, 2002) put great effort into identifying the noteworthy events and responding critically to them. In other words, what teachers' notice regarding students' thinking is important to orchestrate the classroom environment effectively. Keeping in mind that teacher knowledge has an important role in the interpretation of students' work and "in the moment" decisions made during teaching, the research reported in this paper seeks to shed light on how prospective teachers' knowledge plays a role in their noticing of students' thinking after the prospective teachers' engagement in a methods of teaching mathematics course.

Teacher Knowledge

There is no doubt that one of the pivotal teacher competencies for effective mathematics teaching is having solid mathematical knowledge, including comprehensive understanding of the content and the ability to unpack it for the students (Ball et al., 2008). Among the various ways in which teacher knowledge has been categorised, in this study, we are inspired by the Mathematics Teacher's Specialised Knowledge (MTSK) model of Carrillo-Yañez et al. (2018). The reason for grounding the current study on MTSK is to be able to analyse teachers' specialised knowledge from the point of "comprehension and interpretation rather than evaluation" (Carrillo-Yañez et al., 2018, p. 237). The MTSK model focuses on teachers' knowledge and sub-components of knowledge, and interactions between them (Figure 1). The model is comprised of two main domains named Mathematical Knowledge, and Pedagogical Content Knowledge (Carrillo-Yañez et al., 2018). Mathematical Knowledge is divided into three sub-domains: Knowledge of Topics (KoT), Knowledge of the Structure of Mathematics (KSM) and Knowledge of Practices in Mathematics (KPM). Pedagogical Content Knowledge (PCK), which was first identified by Shulman (1986), is divided into three sub-domains, Knowledge of Mathematics Teaching (KMT), Knowledge of Features of Learning Mathematics (KFLM), and Knowledge of Mathematics Learning Standards (KMLS). Procedures, definitions, properties, foundations, representations, phenomenology, and applications are held under the KoT sub-domain. The KSM sub-domain acknowledges the connections within mathematics. Lastly, justifying, and making deductions are considered under the KPM dimension. The first sub-domain in in the Pedagogical Content Knowledge domain, KMLS, includes sequencing topics, expected learning outcomes, and curriculum development. Theoretical knowledge, knowledge of sources and materials are held under the sub-domain of KMT. Lastly, the KFLM sub-domain is related to the knowledge about students' thinking. The *Mathematics Teacher's Specialised Knowledge* (MTSK) model as an alternative to other models also includes beliefs regarding mathematics and mathematics teaching.

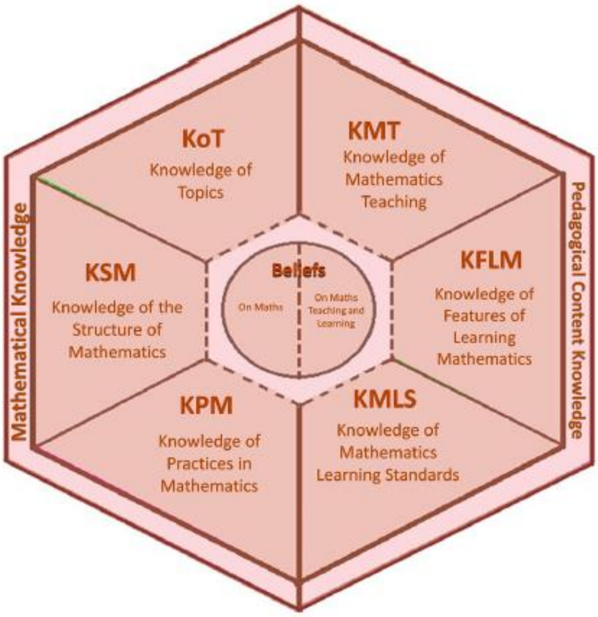

Figure 1. The Mathematics Teacher's Specialised Knowledge model (Carrillo-Yañez et al., 2018, p. 241).

An enduring challenge of research in mathematics education is understanding and explaining teacher knowledge and how it is enacted in instruction. Adler and Davis (2006) emphasised that one of the ways that teacher knowledge is enacted is that teachers interpret the mathematical reasoning and thinking of each student and reflect on these interpretations to make in-the-moment decisions. In relation to the enactment of teacher knowledge, Blömeke et al. (2015) proposed a model to unite such knowledge, which is referred to as cognition with performance. In their research, model of competence as a continuum, they claimed that situation-specific skills, that is, perception, interpretation, and decision-making, (PID model) serve as a role to bridge knowledge and teaching practice. Stated differently, it was highlighted that teachers can transfer their knowledge into instruction through identifying and analysing students' thinking and planning their next moves based on students' thinking. Transferring teacher knowledge into instruction has been examined from different point of views. For instance, Ball and Bass (2000) focused on the responsibilities and acts guided by teacher knowledge from the point of noticing to evaluating students' thinking. In addition, Schoenfeld (2011) pointed out that teacher knowledge is a determinant of both the noticing of the noteworthy events during the instruction and the instructional decisions teachers take based on these events. In other words, he asserted that teacher noticing is interrelated with different components of teacher knowledge and does not stand exclusively. Considering the close connection between noticing and knowledge and Yang et al.'s (2021) expression, "the development of teacher noticing is a strongly knowledge-guided process" (p. 816). In this paper, the focus is on the extent to which teacher noticing interplayed with teacher knowledge.

Teacher Noticing

Teacher noticing is defined as the process that encompasses the way in which teachers manage the "blooming, buzzing confusion of sensory data" that they face during instruction (Sherin et al., 2011a, p. 5). During instruction, teachers need to attend to the events that occur, interpret these events, and relate them to similar events that they encountered previously. Star and Strickland (2008) attributed a meaning to teacher noticing as observing and perceiving different aspects of classroom activity such as mathematical content, classroom environment, and teacher and student's interaction. On the other hand, some researchers (e.g., Colestock & Sherin, 2009; Sherin & van Es, 2009) conceptualised teacher noticing as not only focusing on classroom activity but also on the teacher's interpretations of that activity (Colestock & Sherin, 2009; Sherin & van Es, 2009).

National Council for Teachers of Mathematics (NCTM, 2014) emphasised that one of the main tasks of mathematics instruction is the analysis of students' thinking so that teachers can use it as a source for making instructional decisions and improving their practice. Considering the benefits for both students and teachers, Jacobs et al. (2010) integrated children's mathematical thinking into teachers' noticing skills and proposed *Professional Noticing of Children's Mathematical Thinking*, on which the study reported in this paper was grounded.

Professional noticing of children's mathematical thinking

Professional noticing of children's mathematical thinking is comprised of three interrelated skills: "attending to children's strategies, interpreting children's understanding and deciding how to respond on the basis of children's understanding" (Jacobs et al., 2010, p. 172). Jacobs and colleagues built their conceptualisation of teacher noticing how and to what extent teachers notice children's mathematical thinking rather than the diversity of what teachers notice. According to Jacobs et al., while *attending to children's strategies*, the teacher identifies the mathematically significant details of students' strategies/explanations/responses. While *interpreting children's understanding*, the teacher makes sense of how the children conceptualise the mathematics topic/mathematical idea in consistent with the details of their strategy, and while *deciding how to respond* based on children's understanding, the teacher makes instructional decisions based on children's understanding. Furthermore, Jacobs et al. emphasised that the three skills form an interconnected process that leads to creating a classroom environment based on children's understanding. More specifically, attending to children's strategies and interpreting children's understandings are the starting points for deciding how to respond based on children's understandings (Jacobs et al., 2011). However, when a child presents a written strategy or verbal explanation, the three skills must be implemented almost simultaneously, as though forming a single unified teaching move before the teacher responds. In this sense, teachers' noticing should be regarded as an integrated skill.

Student-invented strategies

Any student constructed procedure that does not include the use of physical materials is called an invented strategy (Carpenter et al., 1998). Before the introduction of traditional algorithms, teachers should devote a significant time on student-invented strategies and assist students to build on them before the introduction of standard algorithms (NCTM, 2000). Carroll and Porter (1997) asserted that invented strategies suit better with students' natural tendencies since many students tend to perform operations beginning from the left contrary to the standard algorithm (Kamii et al., 1993). Therefore, mathematics becomes a meaningful activity for students through the invented strategies (Carroll & Porter, 1997). Furthermore, invented strategies influence the development of students' number sense and their understanding of the place-value concept and multidigit operations. This supports students to make fewer computational errors and develop flexibility in using their knowledge in different situations (Carpenter et al., 1998; Carroll & Porter, 1997).

Whole number multiplication, which is the mathematical concept focused on the present study, is regarded as one of the most difficult operations (Flowers et al., 2003). There is, however, agreement in research literature that students can come up with various invented strategies for multiplication operation if they are provided with sufficient time (Baek, 2006; Carroll, 1999). Son et al. (2013) categorised five invented strategies for multiplication as direct modeling, repeated addition, chunking method, compensating (or varying method), and partial products (or decomposition, or partitioning method). With the inclusion of doubling strategy mentioned in Baek (2006), it can be stated that students use six different invented strategies for whole number multiplication in addition to the standard algorithm. In the research, we used partitioning, compensating and doubling strategies. Son et al. (2013) defined the partitioning strategy as "us[ing] the base-ten structure to break down the factors into partial products and use the distributive property", and the compensating strategy as "round[ing] one of the factors to a multiple of 10 to make the multiplication easier and compensate by subtracting the extra" (p. 694). Additionally, the doubling strategy is "continuing pairing [the same number] and figured out the product by adding two final subtotals" (Baek, 2006, p. 244). One of the reasons for selecting these strategies is that students generally prefer to use these strategies (Baek, 2006; Son et al., 2013). Moreover, these strategies do not require a lot of steps; that is, they are relatively efficient. Complex doubling strategies shorten the calculation procedure considerably (Baek, 2006). However, when larger numbers are included, these strategies can lead to errors (Son et al., 2013). Therefore, understanding and evaluating student's invented strategies are the first steps that teachers should take to build student understanding on the concepts targeted (Campbell et al., 1998).

Significance of the study

Jacobs et al. (2010) claimed that expertise in professional noticing of children's mathematical thinking is learnable and develops over time throughout a teacher's career. Considering Jacobs et al.'s study, mathematics educators have started to investigate teachers' noticing of children's mathematical thinking within the context of specific mathematics topics and applied a variety of interventions to improve student skills. For instance, interventions in measurement (Girit-Yildiz et al., 2021), early algebraic reasoning (Fisher et al., 2019), early numeracy (Schack et al., 2013), and geometry (Ulusoy & Cakıroglu, 2020) aimed to explore prospective teachers' noticing of students' thinking via video analysis and discussion sessions. All these studies revealed that prospective teachers provided more details and explanations related to students' thinking after participating in the intervention. In addition, the researchers argued that prospective teachers' noticing skills could be improved with enriched method courses during their teacher education program (Stockero, 2014; Tekin-Sitrava et al., 2021). Drawing on this point of view, we designed an intervention enriched with reasoning about students' thought processes and strategies related to numbers and operations in a method of teaching mathematics course. Thus, we believe that this study contributes to the research on teacher education in how method courses could be implemented to develop prospective teachers' noticing skills.

Moreover, acknowledging the development of noticing skills within teacher education programs, mathematics educators start to examine new methods and conditions, which might ensure the prospective teachers improve their knowledge and noticing skills to teach the mathematics effectively (Blömeke et al., 2014; Henry et al. 2014). Accordingly, in recent years, many researchers investigated the intersection between teacher knowledge and their noticing skills (e.g., Fisher et al., 2018). Although knowledge has an outstanding role in noticing children's mathematical thinking, the interplay between them is still an unanswered question and needs to be investigated. To address this gap, in the research implemented, we provided an intervention to enhance a prospective teacher's noticing skills and examined evidence of the knowledge that supported the interplay between knowledge and noticing that emerged during the process.

Finally, many researchers investigated the relationship between knowledge and noticing skills by grounding their work in the Mathematical Knowledge for Teaching framework (e.g., Fisher et al., 2017; Jong et al., 2021). But, different from these studies, in this study, prospective teachers' knowledge was explored within the frame of the dimensions of MTSK due to analysing knowledge from the point of "comprehension and interpretation rather than evaluation" (Carrillo-Yañez et al., 2018, p. 237). Moreover, the related studies did not explore which type of knowledge nested which type of noticing skill. In reporting the outcomes of the study, we aim to provide detailed analysis related to which knowledge types nourished which noticing skills in the context of whole number multiplication. Thus, the following research questions guided the study:

1. *How does a prospective teacher's attention to and making sense of student's invented strategies, and the decisions made in responding to those strategies within the context of whole number multiplication change after the prospective teacher's involvement in the instruction enriched with reasoning about students' thought processes and strategies?*
2. *Which knowledge evidence underpins these changes that support the interplay between knowledge and noticing?*

Method

A qualitative case study was employed to provide in-depth exploration of the change in one prospective teacher's noticing skills and the knowledge evidence underlying this change. There are several reasons to choose case study design as a methodological approach. Case study is a trustworthy guide and allows complex phenomenon to be understood and explored where holistic and in-depth analysis is required (Merriam, 1998). The aim of this study is to investigate how a prospective teacher's knowledge plays a role in the change of his noticing skills after his involvement in the instruction in detail and holistically. Also, as Yin (2009) stated, the case study is one of the qualitative methods that seeks to answer how and why questions in situations where the researcher does not control the variables. In the current study, the variables, teacher's noticing skills and his knowledge, were not controlled and the question, how a prospective teacher's knowledge supported his noticing skills change after his involvement in the instruction. Furthermore, the study aimed to describe a phenomenon within the case boundary (Creswell, 2011), that is the study was bounded by noticing skills of an Individual prospective teacher. Among the various types of case study, specifically, the current study is the instrumental case study in which the aim is to provide a general understanding of the issue using a special case (Stake, 1995).

Context of the Study

This study was conducted in the context of an Elementary Mathematics Education (EME) program under the Mathematics and Science Education department at a public university. The language of education in the university is English. Prospective teachers graduating from this four-year undergraduate program are qualified to work as middle school mathematics teachers, responsible for teaching mathematics at Grades 5 to 8 (children aged 10–14 years).

The program offers prospective teachers four types of courses related to mathematics: mathematics education, pedagogy, and common compulsory courses like English for Academic Purposes I. Whereas the prospective teachers complete the courses about mathematics during the first two years, the program focuses on the pedagogy and mathematics education courses in the subsequent years. The program's last year includes the school experience and the practice teaching courses. In addition to compulsory courses, the prospective teachers should complete six elective courses. While three of those courses should be related to mathematics education, such as Exploring Geometry with Dynamic Geometry Applications, the prospective teachers may take the other three courses from any department in the university according to their choices.

Participant

In this study, we focused purposively on one prospective teacher, Matt, who participated in the intervention enriched with reasoning about students' thought processes and strategies related to numbers and operations in the methods of teaching mathematics course. Matt was a third-grade mathematics education student enrolled in the EME program described above. At the time of data collection, Matt took the last mathematics courses offered in the program, Linear Algebra and Methods of Teaching Mathematics I as his must courses. He successfully completed all mathematics courses presented in the first two years of the program. Moreover, Matt stated that he did not have any teaching experience in school classrooms.

Intervention

The intervention was an instruction enriched with reasoning about students' thought processes and strategies. Several studies indicated that videos from diagnostic interviews with students and students' written works were helpful for the development of prospective teachers' noticing skills (Jacobs et al., 2010; Fisher et al., 2018). Therefore, we integrated at least one video or students' written work into our instruction for each part of the course in Table 1. Six videos (see Table 1), five of which were from the Cognitively Guided Instruction (CGI) research program (Carpenter et al., 1996), and a significant number of students' written artifacts were presented to the prospective teachers during the discussion of each part In the videos, a teacher presented a word problem to a student and asked him to solve it without giving any direction. After solving the problem, the student explained his/her solution strategy. During this process, materials were provided to the student if the student wanted to use. In three videos, students could solve the word problems asked through addition or subtraction operations. The other half of the problems required the students to do either multiplication or division operations. Specifically, while two students performed multiplication operations to solve the problems asked, one performed the division operation. From the written artifacts, a student's invented strategy was provided to the prospective teachers to analyse. To more clearly indicate how the intervention was carried out, the last part of the course (2.5 in Table 1) is given as an example.

Table 1
Data Collection

Pre-assessment	Noticing pre-test (applied to 23 prospective teachers)
Intervention (the instruction enriched with reasoning about students' thought processes and strategies)	Part of the methods of teaching mathematics course related with numbers and operations 2.1. Early number concepts and number sense 2.2. Meaning of four operations - A video about addition & subtraction operations - Two videos about multiplication & division operations 2.3 Basic facts for four operations 2.4 Whole number place value concepts - A video about addition & subtraction operations 2.5 Teaching of four operations with whole numbers 2.5.1. Direct modeling of the operations 2.5.2. Student-invented strategies - A video about addition & subtraction operations - A video about multiplication & division operations 2.5.3. Standard algorithm for the operations
Post-assessment	Noticing post-test (applied to 23 prospective teachers) and semi-structured interview (applied to only Matt)

The prospective teachers first modeled four operations together with their instructors through manipulatives. After the discussion of what an invented strategy is, different invented strategies were provided for the four operations. For example, a student's written work for the multiplication operation of 35×12 was shown to the prospective teachers. In this work, to multiply 35 with 12, Kenneth takes the first multiplicand apart, and deals with easier multiplication operations, 10 × 12 and 5 × 12. After he decomposed 35 as three tens and one five, he added the result of 10 × 12, 120, three times and finds 360. In the last step, by adding 60, which is the result of 5 × 12, to 360, he reached the correct result, 420 (Van de Walle et al., 2013). Lastly, the standard algorithm was explained by the instructor and modelled by prospective teachers through manipulatives. Whether and how the previously discussed invented strategies were related with the standard algorithm were also discussed with prospective teachers.

Formal training regarding noticing at the beginning of the intervention was not provided. However, three prompts about three interrelated skills of professional noticing of children's mathematical thinking were asked to the prospective teachers after each video or student work, respectively and a large group discussion occurred around these prompts. For instance, the instructor provided the example student work explained above to the prospective teachers and firstly gave them some time to let them prepare for the discussion. More specifically, she asked them to think about Kenneth's strategy through such questions: "How did this student find the solution to 35 x 12? Whether his strategy is correct or not? Does his strategy work every time and when is it better to use"?" Subsequently, a whole class discussion occurred around these questions to identify the student's strategy in detail. After attending to Kenneth's strategy, the instructor asked what his strategy implied about his understanding of the multiplication operation. At the end of a similar whole class discussion, the instructor posed some questions about the deciding skill. That is, she requested their next steps for Kenneth and their reasoning. During the whole class discussion, suggestions of the prospective teachers were compared with each other through such questions: What do you think about X's recommendation? Should we use it before or after the Y's example? What does this provide to Kenneth? The same process was performed after each video and students' written works.

As a final experience, as in the study of Fisher et al. (2018), prospective teachers were assigned to conduct a diagnostic interview with a student about one of the topics in Table 1. To complete their assignment, they were to analyse the student's understanding by using the guiding questions related to the professional noticing framework (Jacobs et al., 2010).

Data Collection

Pre- and post-noticing tests

Students' written work serves as a window to track students' mathematical understanding (Carpenter et al., 2003). Therefore, the noticing tests were composed of hypothetical students' written work involving their invented strategies regarding whole number multiplication. In terms of students' invented strategies, we concentrated on three strategies regarding whole number multiplication: partitioning, compensating, and doubling. Furthermore, by taking into consideration that interpretations and responses of teachers may vary based on the correctness of the students' answers (Son et al., 2013), the noticing tests, applied to 23 prospective teachers, involved both correct and incorrect students' works, which are presented in Appendix A.

The first student's work, Sude, was prepared by the researchers. In this solution, Sude tried to solve the operation 38 × 6 with the strategy of partitioning. However, she failed to apply multiplying both parts with the second factor. Sefa's work was taken directly from the study of Baek (2006), who named this strategy as complex doubling. In addition to being a very efficient strategy by decreasing calculations, complex doubling strategy indicates to have an advanced perception of the unit as a number concept (Baek, 2006). The other student's work, named Eda, was based on the compensating strategy, which is one of the most frequently used strategies by students (Son et al., 2013). This solution was prepared by the researchers. In this solution, Eda solved the operation 25 × 16 by multiplying 16 with 100 at first. Then, she divided the result, 1600, into four.

While the works of Sude and Sefa were used as a pre-test, the works of Sude and Eda were as a post-test . More specifically while designing pre and post-test, we paid attention to correct and incorrect student solutions. The reason for using Sude's solution in both pre-test and post-test was to evaluate prospective teacher's noticing skills on student thinking persistent common misconceptions. Sefa's and Eda's solutions, however, were assigned to the pre-test and post-test respectively as correct solutions. The reason for applying different correct solutions in the pre-test and post-test was that both solutions involved different mathematical details related to the multiplication with whole numbers. By this way, it was aimed to evaluate to extent to which the prospective teacher can identify different mathematical details embedded in different solutions and to what extent he could make sense of student understanding that led him to develop such a solution.

After examining each students' work, Matt was requested to answer four questions related to the three interconnected skills of noticing. As seen in the Appendix A, the first two questions were related to attending and interpreting skills and the other two questions addressed the skill of deciding. More specifically, the aim of asking the first two questions was to determine the extent to which Matt explained the mathematically important details embedded in students' solutions and make sense of a students' understanding. The goal was to examine Matt's in-the-moment decisions regarding students' understanding demonstrated by answering the third and fourth questions.

Data collection procedure

The data collection process involved three phases, which are summarized in Table 1. In the first phase (pre-assessment), 23 prospective teachers completed the first noticing test as a pre-test before the intervention. Subsequently, in the second phase (intervention), they were involved in a 5-week intervention about numbers and operations, which is explained in the following section in detail. In the third phase, the post-test and interview were applied. More specifically, all 23 prospective teachers were given the second noticing test as a post-test immediately on completion of the intervention. After the analysis of the prospective teachers' noticing levels according to the framework (see Appendix B), Matt was selected among the 23 prospective teachers for a semi-structured interview (Fylan, 200) and was asked whether he was voluntarily participating in semi-structured interview for the in-depth exploration. As Merriam (1998) claimed, more specific information not obtained through written tests and not observed could be collected via interview since it gives opportunities to enter participant's mind by asking specific questions related to the issue to be explored. Since he reached the highest level of noticing for each dimension in the framework after the intervention, we had a chance to get more detailed and in-depth insight into prospective teacher's attention to and making sense of student's invented strategies. Once the details of the interview were explained, Matt signed the consent form.

The first aim of this interview was to clarify Matt's responses in both pre and post-tests. The other and major aim was to get in-depth information regarding the knowledge evidence behind his development regarding the second research question. A sample question asked to Matt for this aim was as follows: "Sude applied a different strategy rather than applying the standard algorithm directly. What kind of knowledge may Sude have to invent this strategy?" The aim was to capture Matt's knowledge regarding the distributive rule and place value in his response. The interview lasted approximately 40 minutes and was conducted by two researchers. The interview was video recorded and transcribed verbatim for analysis.

Data Analysis: Pre- and Post-test

Data were analysed from two aspects: professionally noticing of students' mathematical thinking and prospective teachers' mathematical knowledge. To reveal how the prospective teacher's engagement in teaching method course regulated his noticing skills, the data were analysed with respect to the dimensions of the *Professional Noticing of Children's Mathematical Thinking* framework presented by Jacobs et al. (2010). In their framework, three interrelated skills of noticing were evaluated at three levels of evidence, Lack, Limited and Robust. During the analysis of the data for this study, it was realised that although those categories were very comprehensive and detailed, they did not cover all aspects of our data. For instance, we needed one more category between limited and robust evidence to code the data related to attending and interpreting skills more deeply. To meet this need, the categorisation developed by Tekin-Sitrava et al. (2021), which emerged by expanding Jacobs et al. (2010) 's levels, was used as an evaluative framework. Within this framework of categorisation, prospective teachers' attending and interpreting skills were coded as *Lack, Limited, Substantial,* and *Robust Evidence.* In relation to the deciding how to respond skill, a quite different categorisation was needed to reflect Matt's responses more explanatory. In this context, it was coded as *Ignorance, Questioning, Challenging,* and *Responding and incorporating*. Moreover, the features of each category related with each skill were determined through open coding, which was used to determine the properties and dimensions of the data (Strauss & Corbin, 1998), to make the data analysis process more explicit and detailed. For example, to code how the prospective teachers attended to the given student's strategies, we first identified the mathematical significant details like the distributive property, partitioning strategy, or the place value concept. Accordingly, if the prospective teachers explained these details, we coded that *robust evidence* was provided by the teachers. However, if these details were explained but not with an appropriate mathematical language or they presented some of the mathematical details, prospective teachers' comments indicated *substantial evidence* of attention. Besides, if the prospective teachers identified only a few details or they correctly identified the solution as incorrect/correct by using alternative numbers/alternative solution strategy without referring to the present situation, we coded that they provided *limited evidence* of attention to the given students' strategies. Lastly, if the prospective teachers presented an answer independent from the given ones or if most of the mathematical details were missing, then prospective teachers' comments provided *lack of evidence* of attention. The details of the levels for the other skills are presented in Appendix B.

In line with the purpose of the study, the prospective teacher's mathematical knowledge before and after his engagement in the teaching method course was examined by the data gathered from pre-test, post-test and follow-up interviews. The prospective teacher's knowledge was analysed through Mathematics Teacher's Specialised Knowledge (MTSK) model developed by Carrillo-Yañez et al. (2018). In their study, categories of knowledge and descriptors for each sub-domain were presented. For example, the knowledge of features of learning mathematics (KFLM) sub-domain involves four categories, which are *theories of mathematical learning*, *strengths and weaknesses in learning mathematics*, *ways pupils interact with mathematical content, and emotional aspects of learning mathematics*. Accordingly, if the prospective teacher presented some ideas about one or more of these categories, we coded this as evidence of KFLM. Afterwards, to specify the interplay between prospective teacher's knowledge and his noticing skills within the context of whole number multiplication, the levels of noticing skills of prospective teacher before and after the teaching method course were compared. Accordingly, when the knowledge evidence that the prospective teacher had in the post-test data was captured, then it was categorised according to the MTSK theory. For instance, Matt's attending to Sude's solution, which was incorrect, increased after his enrollment to the intervention and the difference between his attending in the pre-test and post-test seemed by explaining the result using the tens and the ones terms, and application of distributive property of multiplication over addition. This showed that he knew the concept of place value and distributive property related to the rules and the procedures. In other words, it could be deduced that Matt was aware of Sude's reasoning while applying the operation. This was regarded as the evidence of KFML.

The entire data analysis process was carried out by the authors of the paper, who are experts in the mathematics education area, to ensure the inter-rater reliability. Inconsistencies identified in the coding were discussed during the data analysis until reaching 100% consensus among the coders. For instance, one author coded Matt's response related to Sude's solution in the pre-test as robust evidence of attending skill. The other authors, however, coded it as substantial because Matt did not mention details of the distributive property of multiplication over addition, and the partitioning by decades strategy. In this instance, the assignment of the code, substantial, was accepted because a discussion by all coders involved determined why the response would not be regarded as robust.

Findings

The aim of this study was to reveal how instruction enriched with reasoning about students' thought processes and strategies regulated a prospective teacher's noticing skills in the context of whole number multiplication. In addition, we examined the knowledge evidence that supported the interplay between knowledge and noticing that emerged during the data analysis process.

Prospective Teacher's Noticing Skills for Incorrect Solution Strategy

Prospective teacher Matt's attending to a student's incorrect solution strategy (Sude's solution), his interpretation of the student's misunderstanding, and his decision on how to respond to the student are summarised in Table 2. In addition to the prospective teacher's noticing skills, knowledge of evidence, which highlights the differences in the noticing skills between pre and post-test, is also added to the table.

Table 2
Pre and Post Findings of Prospective Teacher's Noticing Skills for Incorrect Solution

Noticing Skill	Pre-test	Post-test	Evidence of Knowledge
Attend	Substantial	Robust	KoT, KFLM
Interpret	Substantial	Robust	KoT, KFLM
Decide	Questioning	Responding to and incorporating	KoT, KMT, KMLS

Prospective Teacher's Expertise In Attending to and Interpreting Skills for Incorrect Solution

Matt's response showed that the level of his expertise in attending to a student's incorrect solution strategy and interpreting student's misunderstanding is substantial before the intervention and robust after the intervention. To illustrate, his explanations of the mathematical details in a student's incorrect solution (Sude's solution) and his interpretation of Sude's understanding are as follows.

> Before intervention (Sude's work from pre-test)
>
> Here, Sude does the multiplication operation using distributive property. Namely, she wants to separate 38 and to multiply 30 and 8 by 6. In fact, she sees the multiplication as repeated addition. She wants to add 30 many 6 to 8 many 6. But she forgot to multiply 8 by 6.
>
> After intervention (Sude's work from post-test)
>
> Indeed, she wants to produce a very elegant and easy strategy. Thirty-eight is a large number but she divides this number into tens and ones which will make computation easier. In other words, she wants to use distributive property. Thus, she thinks 38 as 30 and 8, then operation turns into (30+8) x 6. Indeed, here we have tens and ones. She just focuses on multiplication of 3 tens with groups of six (he shows number 6 out of parenthesis). We have six groups of 3 tens and six groups of eight. Thus, we have 180. But she missed that in each group we have 8. Thus, she missed multiplying 8 with 6 and just focused on 8, the number in one group. Then, she directly adds 8 to the product of tens which is 180.

Before the intervention, Matt correctly identified the solution as incorrect and interpreted Sude's understanding based on details of the distributive rule by presenting appropriate mathematical language. However, the details regarding distributive property of multiplication over addition and partitioning by decades strategy were missing in his response. After the intervention, Matt still stressed that the student used the distributive property and had managed the first part (30 x 6) but missed the second part (8 x 6) and then added 8 to the 180. The notable difference in Matt's response is that after the intervention, he paid attention to the place value by stating the tens and the ones terms. Due to this difference, Matt increased his level of attending and interpreting skills from substantial level to robust level. An interview conducted after application of post-test further revealed that he knew the meaning of the concept of place value and distributive property related to the rules and the procedures. In other words, he was competent in conducting an algorithm that was not conventional and stressed the necessary conditions to perform the operation. He stressed the multiplication of six groups with 3 tens and units, which is the direct application of distributive property of multiplication over addition. Parallel to this, in terms of KFLM, we can say that he was aware of the student's understanding of the knowledge of distributive property of multiplication over addition based on the operation performed and commented that the student grappled with the problem because he missed performing the second half of the operation. Thus, we could deduce that Matt was aware of the student's reasoning while performing the operation, which can be accepted as evidence that KFLM emerged during the post interview.

Prospective Teacher's Expertise in Deciding how to Respond Skill for Incorrect Solution

Matt's skill of deciding how to respond based on children's understanding showed an improvement via the intervention. Although before the intervention his level was determined as questioning with respect to the properties of each category of deciding skill presented in Appendix B, after the intervention, it was regarded as responding and incorporating. To illustrate its development, his responses to the student were as follows.

> Before intervention (Sude's work from pre-test)
>
> Firstly, I asked why she multiplied 30 and 6. I want to be sure that she knows distributive property. Then, I asked why [she] added 8 to that product because she made a mistake there. Then, I asked whether she thinks she performed the operation correctly.
>
> After intervention (Sude's work from interview)
>
> Before answering how to respond to Sude, Matt read his response, which he wrote after the intervention.
>
> Interviewer: Could you elaborate, what [do] you mean by another method?
>
> Matt: Like using estimation. I want her to use the rounding method and perform the operation. For instance, 38 can be rounded to 40. 40 times six is equal to 240. But 188 and 240 are really far away. She just increases the number by 2 but the difference is too much. Then, she might realise her mistake. I can also ask to use division to check her answer if she has learnt the division operation. I mean multiplication and division are inverse operations. But she figures out that 188 cannot be divisible by 6. Thus, she realises that she made a mistake.
>
> Interviewer: If she could not realise that she should multiply 6 with 8, what will you do?
>
> Matt: I will use base ten blocks. I got 228 and let her divide them into 6 equal groups. I mean this time I gave her 6 groups and she should realise that each group has 38 pieces. But 38 x 6 is not 228. I mean multiplication is repeated addition and when she adds all of them up, she could understand her mistake. Lastly, I stressed standard algorithm. I connect the invented strategy with the standard algorithm. I mean multiply ones with ones and add carrier to the multiplication of tens by tens. In fact, we apply distribution rule while performing standard algorithm and standard algorithm is the easiest way of multiplying the numbers which have at least 2 digits.

Before the intervention, it was obvious that Matt planned to make the student realise her mistake through critical questions. After the intervention, Matt continued asking critical questions to support her to realise her mistakes. However, he further elaborated the student's solution through alternative methods. More specifically, as it could be realised from the excerpts of the interview, Matt stated that he would use alternative strategies, materials, or models/approach such as estimation and using base-ten blocks. Moreover, he intended to incorporate further understanding by presenting how to apply a standard algorithm with a rationale. In this way, he aimed to make a generalisation regarding the multiplication operation. Accordingly, it could be concluded that Matt's instructional decisions as the next teaching moves developed from questioning to responding and incorporating level after the intervention. Based on the dialogue above, there is evidence of KoT referring to the knowledge of the procedures of standard algorithms. Matt stressed the conventional way (standard algorithm) to conduct the multiplication operation where he directly multiplied ones (8) with ones (6) and add the carrier to the tens place and then multiply ones (6) with tens (3) to let the student check her answer. In addition, while referring to the inverse operation, he stressed whether they know the division or not which can be accepted as indications of KMLS, which involves the knowledge of mathematical content to be taught, the sequencing topics, and determining the expected learning outcomes. Another knowledge evidence that is assumed to be developed during the intervention is KMT. During the post interview, it was obvious that Matt was aware of potential strategies (e.g., using estimation) and manipulatives (e.g., based ten blocks) for teaching multiplication of whole numbers.

Prospective Teacher's Noticing Skills for Correct Solution Strategy

The prospective teacher's noticing skills for correct solutions (Sefa's and Eda's solution) were categorised to disclose how prospective teacher's knowledge regulated his noticing skills within whole number multiplication. Matt's expertise in all three noticing skills before and after intervention is summarised in Table 3. Moreover, the knowledge of evidence that caused the differences in Matt's noticing skills between pre- and post-intervention is also presented in the table.

Table 3
Pre and Post Findings of Prospective Teacher's Noticing Skills for Correct Solution

Noticing Skill	Pre-test	Post-test	Evidence of Knowledge
Attend	Robust	Robust	KoT, KMT
Interpret	Limited	Substantial	KoT, KFLM
Decide	Questioning	Challenging	KFLM, KMT

Prospective Teacher's Expertise in Attending and Interpreting Skills for Correct Solution

Based on the properties of each category of attending skill presented in Appendix B, Matt's attending to the student's strategy was robust before and after the intervention. From the point of interpreting the student's understanding in consistent with the details of their solutions, Matt's level of interpreting skill is regarded as limited before and substantial after the intervention. His explanation of the solutions and interpreting students' understanding are presented below.

Before intervention (Sefa's work from pre-test)

> Sefa is aware that multiplication is repeated addition. First, he doubled 32, then doubled the result (64), which is equal to 4 times 32 (showing 128 on the paper). On the right-hand side, he writes how many 32s he used. After four attempts of doubling, she got 16 many 32s. Then, he realises that he cannot double anymore and adds the previous solution, eight 32s to the result. He had 16 many 32s before and now in total, when he adds 8 more 32s, he gets 24 many 32s. His answer is right.

After intervention (Eda's work from post-test)

> Eda replaced the multiplication operation 25 x 16 as 100 x 16; she replaces 25 as 100. She thinks it is better to use 100 because we have 25s in 100. Thus, she uses 100 because she can multiply by 100 easily and then could make necessary modifications. To multiply any number by 100 means that you have hundreds as many as that number. Or, it means add two zeros at the end. Student generates this strategy. Since this was easy way for Eda, she developed this kind of strategy. She states that 100 x 16 is 1600. Then she divides 1600 by 4 since 25 x 16 is being asked. And we have 4 many 25s in 100. She multiplied 25 by 4 at the beginning, and then she divides 100 by 4 at the end. In other words, she makes the number smaller which she makes bigger at the beginning. When we look at right hand side, she reaches 16 by counting 4s since we have 4 many 25s in 100. She also checks her answer as 4, 8, 12, and 16. Her solution is correct then.

Matt explained the details of the solution step by step correctly before and after the intervention. Regarding Sefa's work, he identified how Sefa used the doubling strategy and then expressed why he ended up using the doubling strategy and adding 8 more 32s. Matt identified Eda's work, which depended on a compensating strategy, by stating the reason for multiplication and division by 4 respectively. Also, Matt attended to the way Eda checked the solution. Due to Matt attending to both students' solutions with detailed information, his expertise in attending skills before and after the intervention was determined to be robust.

Although Matt explained the solution using repeated addition and the doubling strategy before the intervention, identification of the associative and distributive rule was missing. Matt did not interpret how Sefa applied the distributive and associative rules while creating new units. He only explained how the student reached 512 by adding 8 lots of 24. Moreover, there was a deficiency in the mathematical language. In other words, while Matt was interpreting the student's correct solution strategy by using the doubling strategy, the details of the doubling strategy were missing. However, Matt's post-intervention regarding interpretation was regarded as substantial. Namely, he correctly identified the solution as correct and explained the solution by referring to the compensating strategy. He did not, however, use the correct name of the strategy and compatible numbers while mentioning "4 many 25s makes 100." This shows he did not interpret Eda's solution by identifying all mathematical details.

While stating the reason for multiplying 100 during the interview, Matt used the expression of landmark numbers, which refers to "multiples of 10, 100, and occasionally other special numbers, such as multiples of 25" (Van de Walle et al., 2013, p. 206). This demonstrated that he employed more mathematical concepts and the ability to use them while attending to the student's solution. Moreover, he stated the reasons for why Eda multiplied by 100 instead of 25 from the point of her learning. More specifically, he contended that using the notion of landmark numbers was the easiest way for Eda, which showed that he was aware of the strategies for doing the multiplication operation. From this instance, it was seen that Matt gained more KMT, which emerged during the process. Matt commented on the student's capability by stating in the interview that "Eda thought that since 'I enlarged the number, I have to shrink it,' and then she divided (referring to 1600/4 = 400)" and how she performed the operation. By explaining the reason for dividing 1600 and 4, Matt interpreted Eda's understanding as related to multiplicative thinking. This instance was accepted as evidence of KFLM referring to "how students think and construct knowledge while tackling mathematical activities and tasks" (Carrillo-Yañez et al. 2018, p. 246).

Prospective Teacher's Expertise in Deciding how to Respond Skill for Correct Solution

While Matt provided evidence for expertise in deciding how to respond at the level of questioning before the intervention, he developed his skill of deciding through the intervention and rose to the challenging level. To Illustrate his skill of deciding, Matt's explanation included:

> Before intervention (Sefa's work from pre-test)
>
> I let him explain his strategy. I would ask why he added 32 to 32 and 64 to 64. Is it coincidence or does he really know the strategy? First of all, I should understand that. Then, I would want him to explain why he stopped at the 5th step, and why he did not add 512 to 512 and he added 256. Lastly, I would ask him which property of multiplication he used while adding 16 many 32s and 8 many 32s.
>
> After intervention (Eda's work from interview)
>
> I may ask division operation but since Eda used landmark, it might create a misconception. I asked the meaning of 25s. I mean does she want to reach 100? Why did she divide 1600 into 4? We have 4 many 25s in 100. Is this the reason for that? I asked her. I also asked is there any other method to do this or could you use a standard algorithm to solve it. I also asked her to use base ten blocks. By using models, she tries to figure out 16 many 25s. She could join 4 many 25s and obtain one 100. By this way I could let her to find the relationship between invented strategy and the standard algorithm.

As a next instructional step, before the intervention, Matt decided to respond to Sefa in the way of making him think about his solution through simple questions based on the steps of his solution. The questions that Matt preferred to ask did not have the quality to extend Sefa's understanding. Instead, the questions resulted in the student doing little more than re-explain his solution. After the intervention, both the data from the post-test and interview showed that Matt encouraged the student to solve the same operation using materials (base ten blocks) and different strategies (standard algorithm). In addition, he could make curricular connections to division and multiplication, which changed the questioning into challenging response types in terms of deciding how to respond.

Thus, it can be seen from his responses after the intervention that Matt was aware of the potential strategies such as the standard algorithm and knowledge of how to use the materials (base-ten blocks) that helped the student extend her understanding. This can be accepted as evidence of the Knowledge of Mathematics Teaching (KMT). Moreover, Matt presented a next move, which was using the division algorithm, but after a while he decided that it might cause a misconception. More specifically, his potential response conveyed his awareness of students' possible difficulties, which is one of the properties of the Knowledge of Features of Learning Mathematics (KFLM).

Discussion

Based on the analysis of the data, the prospective teacher (Matt) developed his three interrelated noticing skills for both correct and incorrect solution strategies through the intervention implemented in method courses. Findings also revealed that both mathematical knowledge and pedagogical content knowledge (PCK) developed and supported the prospective teacher's' noticing skills. This confirms the results of several studies, which concluded that teacher noticing skills are situated in the context of knowledge (Fisher et al., 2018; Schoenfeld 2011). Many studies (e.g., Dreher & Kuntze, 2015; Flake, 2014; Jong et al., 2021), however, did not pay attention on the type of knowledge evidence needed to support the development of each noticing skill. As a contribution to the field of mathematics education research, the interplay between each noticing skill and knowledge types are discussed separately in the following sections.

The Interplay Between Attending and Interpreting Skills, and Knowledge

As it was stated in the descriptions of attending and interpreting skills, teachers need to know the knowledge of procedures, rules, theorems, and their meanings, which the students are expected to learn (Jacobs et al., 2010) as well as knowing "how students think and construct knowledge while tackling mathematical activities and tasks" (Carrillo-Yañez et al. 2018, p. 246). Those domains are in line with the characteristics of KoT and KFLM of the framework of MTSK presented by Carrillo-Yañez et al. (2018). By considering definitions of components of knowledge and noticing skills, it seems that attending and interpreting skills for both correct and incorrect solution strategies have a mutual relationship with KoT and KFLM that could be accepted as the strong interplay between them. Apart from KoT and KFML, the prospective teacher provided KMT as evidence in developing his attending skill for correct solution strategy. This shows that gaining (KMT) had an influence on identifying which strategy that the student used to perform the multiplication operation correctly and the reasons for implementing that strategy. From this point of view, we could say that prospective teacher would benefit from developing a repertoire of teaching strategies to attend to students' correct solution strategies. Thus, instruction focusing on diagnostic interviews with students, students' written works, and a large group discussion has the potential to promote prospective teachers' KoT, KMT and KFLM.

The Interplay Between Deciding Skill and Knowledge

From the point of deciding how to respond skill, the analysis of the data gathered after the intervention revealed that the prospective teacher aimed to extend the student's understanding rather than to make him re-explain his solution. In line with this result, it could be concluded that the knowledge provided to the prospective teacher during the intervention nourished his deciding how to respond skill. The remarkable point here is that the research presented the specific type of knowledge evidences the instructors provided for the development of deciding skill. Parallel to our findings, Barker et al. (2019) emphasised that teachers utilise various types of knowledge while making decisions. More specifically, after the intervention, the prospective teacher showed evidences of KMT, KoT and KMLS while deciding his instructional teaching moves to respond to the students. Moreover, knowledge of students' thinking that the prospective teacher acquired during the intervention strengthened the teacher's deciding skill and thus, the level of the prospective teacher's deciding skill increased. Thus, we could deduce that the knowledge instances that were represented could be accepted as the evidence of interplay between knowledge and noticing. This implies that the intervention nurtured the interaction and development of prospective teachers noticing skills and knowledge.

The other contribution of the present study to the mathematics education research field is related to teacher noticing of student invented strategies. The results demonstrated that the prospective teacher gained more expertise in the noticing of both incorrect and correct student's invented strategies through engagement in the intervention. Although the strategies, which were presented to the prospective teacher in the pre-intervention, are the most common invented strategies and require fundamental understanding (Carpenter et al., 1992), the prospective teacher could not identify all the mathematical details of these strategies before the intervention. The reason for this might be that the prospective teacher may not have been familiar with those strategies and could not envisage the kinds of understanding that the students possessed. During the intervention, however, the prospective teacher had a chance to focus on student thinking, to discuss about how to help students to support their misunderstanding before introducing standard algorithms, and how to extend their correct understanding. This may have enabled the prospective teacher to gain knowledge about the multiplication algorithm, which in turn would nurture his noticing of students' thinking.

In conclusion, as it was assumed theoretically and found empirically that teacher noticing skills for both correct and incorrect student-invented strategies can be changed positively after involvement in a teaching of methods course designed for that purpose. This study presents valuable findings in relation to how knowledge is situated in a teacher's noticing skills in the context of multiplication operation, but there is still need more to do. The future studies could be conducted to reveal the changes in both prospective teachers and in-service teachers' noticing skill in the context of different mathematics domains and based in international contexts. In this way, a more detailed and broader picture of the relationship between knowledge sub-domains and noticing skills could be presented, compared, and contrasted.

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Ethics Declarations

Ethical approval

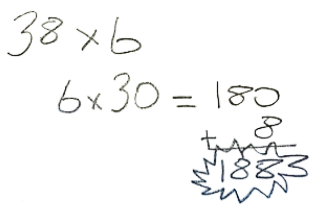
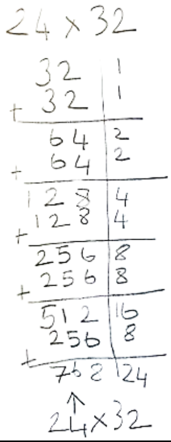
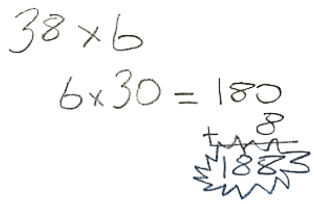
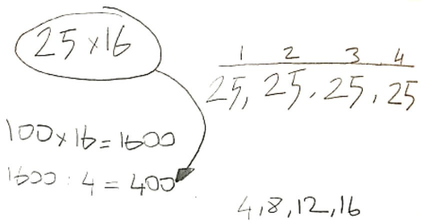
Ethical approval for the research was granted by the Human Research Ethics Committee of Middle School Technical University (MTEU) and informed consent was given by all participants for their data to be published.

Competing interests

The authors declare there are no competing interests.

Appendix A

Students' Works in the Noticing Pre-test and Post-test

Pre-test	**Sude's Work:** 	**Sefa's Work:** 
Post-test	**Sude's Work:** 	**Eda's Work** 

Appendix B

The Details of Levels and Skills of Professional Noticing of Children's Mathematical Thinking Framework

ATTENDING

Lack of Evidence of Attention to Children's Strategies

- Identifying the solution/mathematical concepts correctly but independent from student answer/question.
- Identifying the solution correctly but the mathematical concepts are missing/mathematical language is incorrect/or the explanation includes general statement.

Limited Evidence of Attention to Children's Strategies

- Correctly identifying the solution as incorrect/correct, but there are some naïve conceptions (just mentioning distributive rule, place value without reasoning, repeated addition) and misconceptions while describing the students' multiplication operation.
- Correctly identifying the solution as incorrect/correct by using alternative numbers/alternative solution strategy without referring to the present situation.

Substantial Evidence of Attention to Children's Strategies

- Correctly identifying the solution as incorrect/correct but mathematical language is incompatible and there is some substantial detail but some of the details such as using distributive rule, using partitioning by decades, compensating strategy, doubling strategy or in what way the solution could be correct is missing without efficient use of language.
- Place value concept is missing.

Robust Evidence of Attention to Children's Strategies

- Correct attention to solution through identifying using distributive rule, using the partitioning by decades strategy, compensating strategy, doubling strategy or in what way the solution could be correct but explanation in detail.
- Place value concept is used.

INTERPRETING

Lack of Evidence of Interpretation of Children's Understanding

- Interpreting the solution/usage of mathematical concepts correctly but independently from student answer.
- General statement (lack of Interpretation of the operation).

Limited Evidence of Interpretation of Children's Understanding

- Consisting of interpretation of general statement such as using distributive rule, using partitioning by decades, repeated addition presented using non-mathematical language (however interpretation of associative rule is missing for correct solution strategy).
- Interpretation of the operation without reasoning using alternative numbers/ alternative solution strategy.

Substantial Evidence of Interpretation of Children's Understanding

- Correctly interpretation of the solution as incorrect/correct, there is some substantial details presented using mathematical language but some of the details such as using distributive rule, associative rule, using partitioning by decades, compensating strategy, doubling strategy is missing.

Robust Evidence of Interpretation of Children's Understanding

- Correct interpretation of solution through identifying all mathematical concepts using mathematical language.

DECIDING

Ignorance

- General response ignoring students' thinking.
- Performing the multiplication operation ignoring traditional curriculum.
- Revisit multiplication operation with easy number/direct explanation of standard algorithm.

Questioning

- Performing the same operation by using different numbers without giving any rationale.
- Make students notice his/her solution through questions.

Challenging

- Making the student think about the operation deeply (involving inverse operation) via question.
- Trying to solve the same operation with different strategies, materials or models/approach.

Responding to child and incorporating

- Incorporating further understanding (e.g., to make some generalisation) with giving any rationale.