Pre-service Teachers' General and Specific Arguments in Real Number Contexts

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A study of pre-service primary school teachers' content knowledge regarding real numbers related to infinity, i.e., division by zero and denseness of the real number line, was conducted at a Swedish university. Data were collected twice during the respondents' teacher education using questionnaires and interviews on both occasions. The data were analysed in terms of traces of concept images, focusing on general and specific arguments in different mathematics contexts. The results show that the pre-service teachers often used conflicting general and specific arguments to explain the same phenomena. Some concrete arguments used for explanations represented other mathematical structures than the ones intended. The results raise questions of the pre-service teachers' awareness of their own capabilities. Implications teacher education programmes are discussed.

Keywords · general and specific arguments · real numbers · mathematics content knowledge · mathematics structures · pre-service teachers

Background and Aim

Pre-service teachers studying to become teachers for school years 4–6 (students aged 10–12) study mathematics in Swedish teacher education. Students often have misunderstandings or conflicting views about core concepts in mathematics, for example, division by zero and real numbers (Juter, 2019). Similar results of students' misunderstandings reported in other countries (Crespo & Nicol, 2006; Katz & Katz, 2010; Vamvakoussi & Vosniadou, 2012; Yopp et al., 2011) show the extent and complexity of the problem. Prior research includes studies of pre-service teachers' and other students' understanding of division by zero with several foci, for example, formal and concrete argumentation (Lajoie & Mura, 1998; Tsamir & Sheffer, 2000), intuitive thinking (Tsamir & Tirosh, 2002), and mathematically versus practically based explanations (Levenson et al., 2010).

A substantial amount of research literature has been produced in this area (Crespo & Nicol, 2006), but the publication rate seems to have diminished considerably in the last ten years. Recurring indicators over decades of a lack of instructional materials on division by zero were reported (Dimmel & Pandiscio, 2020), as well as a lack of research on effects of content knowledge on pre-service teachers' instructional explanations regarding division by zero (Karakus, 2018). Thanheiser et al. (2014) conducted an overview of research of pre-service elementary school teachers' content knowledge on whole numbers and operations. The authors found a shift from pre-service teachers' misconceptions to characteristics of what they know. They recommended future research to focus on pre-service teachers' conceptions and the development over time to help teacher educators support pre-service teachers. The present paper is a contribution in that area.

Division and the real number line are taught at basic levels in school years 4-6. The teaching includes rational numbers, decimal numbers, and division with natural numbers as well as relations between fractions, decimal numbers, and percentage. Coordinate systems with grading of axes, and graphs to express proportional relations are also included in the curriculum. More complicated and abstract issues, however, may be evoked from students' queries. Hence, teachers are required to cope with more advanced mathematics and adjust their arguments to students' proficiencies. Teaching mathematics entails presenting concepts and properties, arguing for what make propositions true and explaining how different propositions are related, in theory and in practice (Shulman, 1986). Moreira and David (2008) point to the necessity of making conflicts between applications of academic mathematics (e.g., mathematics generated by professional mathematicians and mathematics for teaching in schools) explicit to pre-service teachers. This means coping with abstract properties of mathematics concepts and contexts in arguments for understanding and teaching. Most of the pre-service teachers' understandings of the number zero and division by zero in Crespo and Nicol's (2006) study were initially based on flawed reasoning and rules. An intervention inspired the pre-service teachers to clarify their own understandings and to investigate the concepts further on their own to improve their explanations. There were, however, many pedagogical aspects left to consider in the explanations. The authors suggest further studies of pre-service teachers' inclinations to justify their mathematical ideas.

Division by zero and properties of the real number line are related to infinity, a concept that often causes difficulties for learners (Sbaragli, 2006; Tall, 2001; Tall & Tirosh, 2001; Tsamir, 2001). Teachers work with the real number line and operations with real numbers through all years of schooling. This means adapting arguments to students at varying levels. Therefore, it is important for primary school teachers to understand comprehensively aspects of infinity in these mathematics contexts to be able to discern mathematical structures in students' thoughts. Yopp et al. (2011) and Takker and Subramaniam (2019) emphasise the importance of considering students' learning progress when teaching decimal numbers. Takker and Subramaniam particularly argue that teachers need to manage the relationship between general and specific representations to cater to students' learning development. Awareness of students' conceptual understanding is important for pre-service teachers' beliefs about mathematical understanding as Stohlmann et al. (2014) found in a study about fraction division. The pre-service teachers reflected on solutions from students with procedural and conceptual understanding. The majority of the pre-service teachers' beliefs changed to regarding conceptual understanding more valuable than procedural for learning mathematics.

Arguments for a mathematical hypothesis may be general or specific depending on the context of the problem, level of abstraction needed to solve the problem, and previous learning experiences. The results from a quantitative study showed that general and specific arguments about concepts related to infinity might lead to opposing outcomes for the same individual (Juter, 2019). Students in teacher education and engineering education responded to a questionnaire with statements about infinite sets, division by zero and denseness of the real number line. Some students responded generally that two different numbers cannot be placed adjacently on the real number line while stating that "1 and 0.99... are different numbers." The responses from many of the pre-service teachers studying to teach students aged 6-12 years indicated that they had fundamental misconceptions. The results did, however, not give any information about the pre-service teachers' arguments for their answers.

Detailed arguments can give explicit information about critical learning issues in mathematics teacher education, for example about generality and mathematical structures. In a study about odd and even numbers, Levenson et al. (2010) found that many elementary school teachers created mathematically based explanations for themselves. They preferred, however, to use practically based explanations for students, believing that students would better understand them. Practically based argumentation requires an accurate mathematical structure for it not to be misleading.

The aim of the research reported in this paper is to deepen understanding of arguments used by pre-service teachers to explain abstract aspects of real numbers at a basic mathematics level during their education. Arguments' *general* and *specific* features and *mathematical structures* are studied. The research questions posed are:

> What types of arguments do pre-service teachers (school years 4-6) use about real numbers in cases of division by zero and denseness of the number line?
>
> How do their arguments develop during three semesters of teacher education?

Theoretical Framework

Content knowledge is "the amount and organization of knowledge per se in the mind of the teacher" (p. 9, Shulman, 1986). This includes structures of concepts as well as strategies for establishing validity of the knowledge. Teachers need strategies to determine whether students' mathematical conclusions and arguments are valid. They hence have to assess the validity of their own mathematics content knowledge to be able to assess their students' learning development. In this paper, individual content knowledge is considered in terms of traces of *concept images* in individuals' actions, such as problem-solving and argumentation. Traces of a concept image are visible, for example, in how an individual solves a task, reasons mathematically, or justifies responses and actions. A concept image is a model for an individual's complete mental representation of a concept, linked to other concepts in a web-like structure (Tall & Vinner, 1981). Learning means creating knowledge by reflecting on new experiences based on existing structures, thus enriching the concept image. Representations in a concept image developed from different contexts can be used simultaneously or be evoked disjointly in different situations. Experiences from different contexts can generate opposing representations in the same concept image and cause cognitive conflicts. Juter (2007) studied students' cognitive conflicts in the case of limits of functions in a calculus course where some students (incorrectly) interpreted a general definition as stating that limits are never attainable. At the same occasion they also (correctly) stated that a specific linear function attained a certain limit value. Different parts of the concept images had been evoked in general and specific contexts. The students' specific knowledge about linear functions stood in conflict with their interpretation of the general definition when evoked at the same time.

There are several words synonymously used in contrast to *general*, for example, *specific* or *particular*. In this paper, the word specific is used in the same sense as in Mason and Pimm's (1984) article to mean certain examples or contexts. This is in line with Font and Contreras' (2008) descriptions of cases of general classes, for example, where a specific linear function, such as $y = 5x - 1$, is seen as a particular case of the general class of linear functions ($y = kx + m$). Specific contexts in this paper include *concrete* features whereas general contexts are *abstract* in their nature. The adjectives concrete and abstract are not regarded as opposites, but as different modes of regarding mathematics items (Coles & Sinclair, 2019). Attention to the items per se is related to concrete modes of activities whereas relations among the items are related to abstract modes of activities. In the study reported in this paper, a specific argument for the hypothesis that one cannot divide a number by zero could be: *If you have four objects to distribute in 0 piles, it is impossible since there is nowhere to place the objects*. This is an argument exemplifying a general rule by attending to items in a concrete specific example. An example of a general argument for the same hypothesis is *Division by zero is not possible due to the relationship between multiplication and division:* $\frac{x}{0} = y \leftrightarrow x = 0 \cdot y$. This argument attends to the relationship between operations on items in a manner generalising to cover all cases. Students' arguments may be held strictly within mathematics, as in the general case about division by zero, or be linked to experiences from the real world with concrete representations, as in the specific case. Font and Contreras (2008) point to the possible challenges of handling specific and general representations, for example, making sure generalisations are correctly justified when using specific generic examples.

Challenges Related to Division by Zero and the Real Number Line

Aspects related to division by zero can be complex and difficult to cope with for many learners as shown in several studies (e.g., Crespo & Nicol, 2006; Dimmel & Pandiscio, 2020; Juter, 2019; Tsamir & Tirosh, 2002). The results imply that pre-service teachers' experiences of division by zero often result in invalid concept images. Many of the pre-service teachers in Crespo and Nicol's (2006) and Juter's (2019) studies thought that the division results in a number, which stand in contrast to Karakus' (2018) findings in a questionnaire study of 197 pre-service teachers studying to teach school years 5-8. Most of the pre-service teachers in Karakus' study responded correctly to six divided by zero. Only one gave a numerical answer. The pre-service teachers in both Karakus' (2018) and Crespo and Nicol's (2006) studies predominately used rule-based arguments. Their responses were nevertheless contradicting, which may stem from different course experiences concerning real numbers. Tsamir (1999) concluded that students tend to adopt all properties from a specific context of numbers when expanding to a more general context, without considering the general conditions. Learners usually start in a context where division always can be performed and only later expand to a context where division sometimes is impossible. This means an expansion to a context including considerations of cases with any real number in the denominator. Tsamir and Tirosh (2002) and Tsamir and Sheffer (2000) stated that many secondary school students thought that division by zero results in a number, as pre-service teachers did in Crespo and Nicol's (2006) and Juter's (2019) studies. The students in these studies failed to consider the circumstances division by zero entails when compared to division by other numbers.

Concrete representations constitute another aspect to consider when making sense of division by zero. Lajoie and Mura (1998) studied pre-service primary school teachers dealing with division by zero. They found that some of the difficulties came from a wish for a physical explanation of the division. The authors stated that concrete representations could hinder understanding with irrelevant elements or limitations. Concrete settings often offer a range of focal points for learning, some possibly interfering with the teachers' intentions. Dimmel and Pandiscio (2020) addressed different learning possibilities in a study of pre-service K-8 grade teachers' handling division by zero using concrete transformable computer diagrams in GeoGebra. Divisions were represented by slopes of lines where division by zero resulted in a line parallel to the y-axis. The aim was to focus on the existing singularity rather than the division that is not possible to perform.

Concepts related to the equality 0.99... = 1 were found to cause problems for in-service fifth-grade teachers' arguments in a study by Yopp et al. (2011). They identified that teachers' fragile understanding of repeating decimals could stem from intertwined senses of number and measurement. The authors also found that teachers keep developing misunderstandings through their careers in relation to their perceptions of their students' learning and other experiences. Many of the students in Juter's (2019) study also had difficulties when dealing with aspects of the real number line. Their responses showed inconsistencies about the denseness of the number line from general and specific contexts when responding to statements about 0.99... and 1. The students considered the contexts separately avoiding the complexity of the inconsistencies, a known strategy for handling complex situations (Tall, 2001). Fischbein (2001) concluded that when individuals handle very abstract concepts, they tend to use alternative representations or mental models, which are more accessible but may lead to contradictions or errors. Fischbein's conclusion, combined with Tall's (2001), that humans often handle complexity by focusing on relevant information in parts, means that students may evoke different parts of their alternative mental models in their efforts to handle abstract or complex concepts. They will then consider local details within a model that is accessible but possibly not entirely accurate.

Methodology

The aim of the research reported in this paper is to contribute knowledge about types of arguments pre-service primary school teachers use about real numbers at a basic mathematics level and how the arguments develop during a part of their teacher education. Methods and participants in the study are outlined in this section.

Students and Settings

A longitudinal study was conducted to investigate pre-service primary school teachers' development to become mathematics teachers. Three groups starting their education in three consecutive semesters took part, 78 students in total. Their beliefs about the mathematics teacher's role, experiences of practice in schools when on practicum, and mathematics content knowledge were studied. Their mathematics content knowledge is reported in this paper. The three groups were taught by the same group of teachers and according to the same curriculum. All pre-service teachers were asked to complete a questionnaire (Q1) during the first week of their first mathematics course (15 from the first group, 30 from the second, and 33 from the third). The first group completed the questionnaire as a pilot study at a lesson where about 50% of the group were missing or did not hand in the questionnaire. Almost all pre-service teachers from the final two groups were present at the time of data collection, and all those present completed the questionnaire (total number of participants: *N* = 78). The questionnaire comprised an invitation to continue as respondents in the study by expressing an interest in being interviewed. Eight volunteered and were individually interviewed (I1) about two weeks after completing the questionnaire (Q1). A preliminary scan of the responses to Q1 and I1, revealed the sample included individuals with various (i) mathematics grades from upper secondary school, (ii) attitudes to mathematics, and (iii) claims about their own mathematical abilities represented in the group (see Table 1). The responses appeared to be comprised of a diverse range of ideas across multiple aspects of the aim of the study; therefore, no further search for volunteers to interview was conducted.

Table 1
Pre-service teachers' mathematics grades from upper secondary school and their claims about their mathematics knowledge from the first sets of interviews and questionnaires

	Mathematics grade/ Attitude to mathematics	Claims about own mathematics
S1	Passed. Mathematics difficult.	Has low self-esteem but is resilient.
S2	High grade. Mathematics easy.	Possesses a lot of mathematics knowledge.
S3	Passed. Mathematics difficult.	Has low self-esteem but is resilient.
S4	High grade. Mathematics easy and fun.	Knows enough mathematics.
S5	Passed. Mathematics difficult and fun.	Needs to learn more mathematics.
S6	High grade. Mathematics interesting.	Knows enough mathematics.
S7	Passed. Positive to mathematics, not interested.	Knows enough mathematics.
S8	Passed. Mathematics easy and fun.	Knows enough mathematics.

Six of the eight pre-service teachers were interviewed individually again (I2), just over a year after I1, in their third semester at the end of their second mathematics course. The other two participants had left the programme and thus the study but agreed for their data from Q1 and I1 to remain in the study. The remaining six completed a second questionnaire (Q2) at the same time as their second interview. Q2 was almost the same questionnaire as used in the questionnaire study to compare students' knowledge about infinite sets, division by zero, and denseness of the number line (Juter, 2019). The study in this paper builds on the questionnaire study (Juter, 2019) with development over time and arguments for responses given the main foci.

The pre-service teachers attended two basic mathematics courses between the two periods of data collection. The first was a five-weeks course about probability, statistics and algebra, and the second a 10-weeks course about number sense, arithmetic and geometry. The real number line was taught in mathematical and mathematics educational contexts on the courses. Empty and marked number lines were used and presented as instruments for teaching. The pre-service teachers studied different types of real numbers, for example, rational numbers in relation to decimal numbers and worked with real-world examples and pure mathematics examples. Repeated decimals and division by numbers between 0 and 1 were parts of the course content. The notation "…" (e.g., 3.14…) for infinitely many decimals was used. Rational numbers were defined as fractions of two whole numbers where the number in the denominator cannot be equal to zero. Division and multiplication were used as inverse operations in arithmetic calculations and algebra. This paper presents the results from the eight interviewed pre-service teachers (called S1–S8). The data revealed different uses of arguments during the main mathematics learning period in their teacher education. The sample is not considered suitable to establish generalisable data to identify large scale trends.

Instruments

A discursive interview methodology (Gobo, 2018) with semi-structured interviews was used in combination with questionnaires. The design with interviews and questionnaires in triangulation (Flick, 2018) was used to collect data suitable for thematic analyses. The questionnaires complemented the interviews allowing respondents to give spoken as well as written responses. The combined methods gave opportunities to validate findings (Bryman, 2016). The first questionnaire (Q1) comprised of openly formulated items, where one concerned division by zero and aspects of real numbers:

> Imagine the following situations and give one or some specific suggestions on how you would explain the mathematics in the situations if you were the students' teacher.
>
> *A*. A student in grade 4 comes and asks you what two divided by zero is.
>
> *B*. Two students in grade 6 discuss how many numbers there are between 0.99 and 1. They cannot come up with an answer and ask you.

The aim was for the pre-service teachers to select their own arguments, general or specific, to explain to the fictitious students in the specific contexts. They were asked to address the mathematics in the questions to avoid vague explanations. The items from the questionnaire and responses to them were discussed in the first interview (I1).

The questions about mathematics in the second set of interviews (I2) were within the same areas as the first. The purpose was to study the content knowledge development after the main part of the mathematics courses in the programme. At the time of the second interview, the six remaining pre-service teachers completed a second questionnaire (Q2) with 14 statements. The words *Agree*, *Do not know* and *Disagree* were listed below each statement, and time was given to mark all their answers before explaining them further. The statements were formulated based on students' challenges in prior research (Crespo & Nicol, 2006; Juter, 2007), experiences of common student responses from teaching, and the pilot study with an earlier version of the questionnaire and interviews to test the questions. The methodological rationale of the questionnaire is described in more detail in Juter (2019). The main aim was to provoke and expose traces of concept images for the concepts rather than have a mathematically exhaustive set of statements about the concepts. This lack of mathematical stringency in some statements may have affected interpretations of some items, which is a possible source of error in the data. However, it was considered more important to have concise and straightforward statements to avoid respondents becoming caught up in complicated formulations. The interviews gave participants opportunities to remedy any interpretation issues. The statements used in this part of the study were (the numbering from Q2 is retained for coherence with other publications):

4. Two divided by zero is a number.
5. Two divided by zero is equal to infinity.
6. You cannot calculate two divided by zero.
7. There is an infinite number of numbers between any two different numbers.
8. There is at least one number between any two different numbers.
9. Two different numbers may lie close together without any other number between them.
10. The number 0.99…, where the number of nines after the comma is infinite, is equal to 1 since there is no number between 0.99… and 1.
11. There are no numbers between 0.99… and 1, but 0.99… and 1 are different numbers anyway.
12. There are numbers between 0.99… and 1, so they are different numbers.

The aim of Statement 4 was to identify whether division by zero was treated in the same way as division by any other real number without recognising the crucial differences in the situations. The following two statements (5 and 6) were selected to reveal how infinity was linked to division by zero (if something could be said to equal infinity or if it is possible to perform the division). In I2, after the pre-service teachers had an opportunity to explain their thoughts about 2/0, they were asked to compare 2/0 with 0/2 for further arguments. The two statements were included to reveal if general rules about division by zero were evoked from a specific context.

Statements 7–9 were formulated to address arguments about aspects of the denseness of the real number line. The following statements (10–12) addressed the same issues in a specific case. Statement 10 was about the specific numbers 0.99… and 1 being equal. Statement 11 was about the numbers being different but close together. Statement 12 was about the numbers being different with other numbers coming between them. The aim was to see how the general and specific parts of the concept images matched when evoked on the same occasion. In the interviews, it was clarified that the statements concerned the numbers on the real number line. The relation between 0.99… and 1 in Q2 was chosen in contrast to the relation between 0.99 and 1 in Q1 to reveal how the pre-service teachers understood the differences in the relations.

Ethical Considerations

The methodology of the study followed the Swedish Research Council's guidelines (Swedish Research Council, 2017) where applicable. The pre-service teachers were introduced to the project at the beginning of their first mathematics course and all were invited to participate. They were informed, orally and in writing, that participation was voluntary and that they could end their participation at any time with no negative consequences. They were also informed that their identities would remain protected from disclosure in all publications and other presentations of the study. The questionnaires were filled out anonymously, except for those who volunteered for interviews. Names and e-mail addresses were in those cases filled out in the questionnaires (Q1) for the purpose of arranging meetings for the interviews. All names were given codes for reporting purposes to protect the pre-service teachers' identities. No sensitive personal data were collected. The researcher worked at the institution where the study was conducted. She did not, however, teach the courses involved in the study or assess any of the pre-service teachers' course work. There were hence no teacher/student power relationships or other relationships that influenced the results. Responses to questionnaires and interviews were not considered as part of the courses in any way and did not contribute to final grades awarded.

Analysis and Results

During the first data collection period, the instruments (Q1 & I1) were used to identify whether the pre-service teachers would use general arguments in a specific context. In contrast, the examples during the second data collection (Q2 & I2) included both specific and general examples. The pre-service teachers' arguments were categorised as general or specific traces of their concept images using theories from Tall and Vinner (1981), Mason and Pimm (1984), and Coles and Sinclair (2019). The arguments were also categorised as correct or incorrect, and the mathematical structures used in the arguments were identified (see Tables 2 and 3).

Arguments Concerning Division by Zero

Responses to the questionnaires (item *a* in Q1 and items 4-6 in Q2) and interview questions regarding division by zero at the two data collection points are presented in Table 2 (with further examples presented in Table 2* in the appendix). Correct means that the example or rule is correct in relation to two divided by zero. It can be an appropriate calculation or a rule applicable for division by zero. The mathematical structures in the arguments are presented in the table. For example, the structure $\frac{2}{1}$ in S2's argument about division by zero in I1 ("You have two of something but nobody to share it with, and you get to keep both yourself" in Table 2* in the appendix). S2 included the individual giving out the objects in the set of people sharing objects, which means the division is by one and not zero as intended.

Table 2
Pre-service teachers' arguments for their answers to what two divided by zero is at the first and second data collection

	First: Q1 and I1, two divided by zero		*Second: Q2 and I2, two divided by zero*	
	General/specific arguments	Mathematical structures	General/specific arguments	Mathematical structures
S1	General incorrect Specific incorrect	$\frac{0}{x}$, $\frac{0}{2}$: Nothing cannot be divided	Specific incorrect	$\frac{2}{0}$=0. Nowhere to place objects
S2	General incorrect Specific incorrect and contradictory	$\frac{x}{0}$= 0. $\frac{0}{2}$: Nothing cannot be divided $\frac{2}{1}$: Keep the items (= 2)	Specific partly correct and contradictory	$\frac{2}{0}$ = 0. The division cannot be done
S3	General incorrect Specific correct	$\frac{0}{x}$: Nothing cannot be divided $\frac{2}{0}$: Nowhere to place objects	-	-
S4	Specific incorrect	$\frac{2}{0}$, $\frac{2}{1}$: Keep the items (= 2) $2 \cdot 0$: Take the items 0 times (= 0)	Specific partly correct and contradictory	$\frac{2}{0}$ = 0. The division cannot be done Nowhere to place objects
S5	Specific incorrect	$\frac{2}{1}$: Keep the items (= 2)	General correct from Specific incorrect examples	$\frac{2}{0}$ = 2. $\frac{2}{1}$: Keep the items $\frac{x}{0}$: Division by 0 cannot be done
S6	Specific incorrect	$\frac{2}{1}$: Keep the items (= 2) change to 2-2: $\frac{2}{0}$ = 0	Specific correct	Compares to $\frac{2}{2}$ and $\frac{2}{1}$ and sees no way to divide 2 by 0
S7	General correct	Division by 0 does not mean anything, hard to explain	-	-
S8	General correct Specific incorrect and contradictory	$\frac{x}{0}$: Division by 0 cannot be done $\frac{10}{0}$: Do not divide with anybody (= 0) $\frac{10}{1}$: Keep the items (= 10)	General correct from Specific incorrect examples Specific unclear	$\frac{x}{0}$: Division by 0 cannot be done Nothing to divide by. Keep the items, possibly $\frac{2}{1}$

Specific and general arguments

The chosen specific arguments had various mathematical structures other than 2/0, that is 2/1, 0/2, 2-2 and 2·0. None of the pre-service teachers explicitly answered the question in relation to their specified examples in Q1. All had the opportunity to explain further in the interviews, and different interpretations appeared. The specific situations revealed misconceptions, such as S6's example from I1 with ships that should disappear to make the division equal to zero (see Table 2* in the appendix). S6 figured out how to correctly handle the division of two by zero by examining specific examples within a mathematical context at I2 after unsuccessfully using the specific real-world example in I1.

Specific arguments often contradicted general arguments. For example, S8 showed evidence of a correct general rule that it is not possible to divide by zero but used specific arguments revealing other mathematical structures and standpoints (Table 2*). The choices of specific examples and explanations revealed that many of the participants did not comprehend division by zero. Specific examples sometimes referred to concrete examples in the arguments. S1 thought it was hard since there was nowhere to place two apples in zero piles, like the responses from S3 and S4. S3 determined, from discussing a specific example about pens, that you cannot divide two by zero. In contrast, S1 and S4, concluded that the division must be equal to zero since it is impossible to distribute two items in zero locales. This shows that the same type of explanation might lead to very different conclusions.

In S4's case, the first answer in Q1 could be interpreted as correct. However, in the first interview, it became clear that he did not see the answer in the non-existing piles of stones, but in the stones still in the hands since it was impossible to place them in piles. Many had this focus on the number of objects to be divided as the result of the division, since there was nowhere to let go of the objects (S1 in I2, S4 in I1, I2, S5 in I1, I2, S6 in Q1 and S8 in I1, I2). Mental concrete representations such as sinking ships (S6) and the distribution of items (S1, S4, S5, S6 and S8) caused problems rather than solving them in most cases in this part of the study. This created obstacles to learning, as Lajoie and Mura (1998) also found in their study. However, S3 and S5 correctly used concrete examples to conclude that it is not possible to divide a number by zero. S5, who had trouble with a concrete example in I1, developed that example in I2 by comparing different mathematical structures. From that she made a correct general conclusion in I2.

The general arguments used had the mathematics structures x/0 and 0/x. There was no confusion with division by 1 or alternative operations, as in the case with specific arguments. Correct general arguments used were about the impossibility of dividing by zero. Incorrect general arguments used were, for example, about zero being regarded as nothing and that it is not possible to divide nothing (in the structure $\frac{0}{x}$).

Content knowledge development

Most pre-service teachers showed an overall improvement in their understanding of division by zero during the time of data collection, as shown in Table 2. However, the majority still had some misconceptions by the time of the second interview. Nevertheless, most had a sense that the division by zero was impossible, which was apparent in the responses at the end of the second interview and in the first interview for S3 and S7, both of whom withdrew from the study after completing Q1 and I1. The majority of the participants changed modes of regarding items in the sense reported by Coles and Sinclair (2019). They went from regarding the objects as concrete entities at the beginning, to regarding the objects in relation to mathematical structures or operations at the end. Some struggled to draw conclusions from their arguments though. S2 thought that the division was not possible (I2) and therefore equal to zero, a position also held by S1 and S4. The problem of making sense of the proposed division resulted in an answer that included a number rather than the conclusion that it was impossible.

The confusion about division by zero was apparent in the data collected. Some, for example S6 (I2), tried to make sense of the division somewhat systematically and then correctly concluded that it was impossible. Others reasoned seemingly unstructured and came to the correct conclusion, such as S8 (I2). S1, S2, S3 and S8 switched the numerator and the denominator and started considering zero divided by two in Q1 and I1. The switch rendered all of them being left with a problem with zero in the new division. S5 had the same problem during the second interview when she was asked to compare $\frac{0}{2}$ with $\frac{2}{0}$. All participants thought that it was not possible to divide zero since zero is nothing or does not exist. The problem of "nothing to divide" in $\frac{0}{2}$, was as big a problem as "nothing to divide by" in $\frac{2}{0}$ for these students.

S2 claimed to have never met the question of division by zero, and was surprised that it was not possible, which is in line with his answer in Q2 (the division can be calculated). S2 went from a general statement that anything divided by zero equals zero in Q1, to a situation in I1 where nothing divided by two could not be done. Then, also in I1, he attempted to exemplify two divided by zero, but ended up with an example of two divided by one. In I2, he similarly claimed that two divided by zero means that two is not divided by anything and therefore equals two. This is the same result as if two were divided by one. Instead of dividing with zero, he interpreted it as not doing anything with the subject of the division. Then he concluded that two divided by zero cannot be done and is thus equal to zero. The sense of not doing anything with the numerator also appeared in the responses from S4, S5, S6 and S8, which is the same as dividing by one.

S8 gave various incoherent answers. During I1 when he said that it is not possible to divide by zero and the result would, therefore, be 'nothing' (unclear what he meant by 'nothing'). At the same time, he compared the situation to a division with yourself (by one), which is possible. S6 started to consider two divided by zero as two divided by one, then showed traces of an incoherent concept image where two divided by zero equals two and then zero. Later he showed uncertainty in Q2 and used examples from within mathematics in I2. When he considered dividing two by zero, he did not come up with an answer. He found no logical way to do such a calculation when looking at patterns. From this, he correctly concluded that it was impossible. All pre-service teachers apart from S1 knew or had a feeling that it was impossible to divide two by zero at the end of the second interview.

Arguments Concerning the Real Number Line

Responses to questionnaires (item *b* in Q1 and items 7–12 in Q2) and interviews regarding numbers on the real number line during the two stages of data collection are presented in Table 3. This table indicates if the responses were correct and what type of arguments were used. More details and examples are presented in Table 3* in the appendix.

Table 3
Pre-service teachers' arguments for their answers to how many numbers there are between 0.99 and 1 during the first data collection and whether 0.99... = 1 in relation to general rules about different numbers during the second data collection.

	First: Q1 and I1, 0.99 and 1		Second: Q2 and I2, 0.99... and 1	
	General/specific arguments	Mathematical structures	General/specific arguments	Mathematical structures
S1	General No answer	Zoom in on the number line Add decimals to 0.99	General correct Specific incorrect	Numbers between 0.99... and 1 The infinite recitation ends 0.99...≠ 1
S2	Specific No answer	Add decimals to 0.99	General correct Specific incorrect and contradictory	Contradicting on numbers between 0.99... and 1 Can add a 9 in 0.99... 0.99...≠ 1
S3	Specific incorrect No answer	0.99 is the step before 1 Compare with %	-	-
S4	Specific No answer	Add decimals to 0.99	General correct Specific incorrect	No numbers between 0.99... and 1 0.99...≠ 1
S5	Specific correct	Add decimals to 0.99	General correct Specific incorrect	No numbers between 0.99... and 1 Can add a 9 in 0.99... 0.99... ≠ 1
S6	Specific incorrect	1-0.99=0.01 so one hundredth between	General unsure Specific incorrect and contradictory No answer	No numbers between 0.99... and 1
S7	General correct	Infinite number of decimals	-	-
S8	Unclear answer	Different combinations	General unsure Specific incorrect	Unsure if there are numbers between 0.99... and 1 1 is whole 0.99... is incomplete 0.99...≠ 1

Specific and general arguments

Some individual arguments for what constituted equal and different numbers conflicted in general and specific contexts (as shown in Table 3). For S4 and S5, the general rule (there are other numbers between different numbers on a real number line) did not correspond with the specific case (0.99… compared to 1). They both stated that there are no numbers between 0.99… and 1, but still considered them to be different. S1 and S2, who shared S4's and S5's general standpoint, stated that there are numbers between 0.99… and 1 and that 0.99… and 1 were not equal. The fact that they could not find such numbers did not make them change their minds. The conflict between the general and specific contexts was possibly not apparent to S1 and S2 because they stated that there were numbers between different numbers in both the general and the specific cases.

S2 and S6 both used contradicting specific arguments. S6 stated that there are no numbers between 0.99… and 1, and that 0.99… and 1 are both equal and not equal. S2 stated that 0.99… and 1 were not equal but was ambivalent concerning the existence of numbers between 0.99… and 1. Their specific arguments stand in contrast to their shared general position that there are infinitely many numbers between any two different numbers. Their general knowledge did not help them resolve their contradictions. S6 was unsure if any two different numbers can be placed tightly together on the number line, and this uncertainty was apparent in the specific case as well. Both S2 and S6 used the lining up of decimals as a specific argument to say that you can always add a 9 in 0.99…. This interpretation means that at some point, the infinite line-up stops so you can add a new decimal, hence creating a new number placed between 0.99… and 1.

Most correct general arguments were about the denseness of the real number line in terms of other numbers between different numbers. The general arguments correctly contradicted the specific arguments, in most conflicting cases.

Content knowledge development

During the first data collection, there were many responses about solving the problem without providing a solution, like the case with division by zero. The pre-service teachers were more prone to formulate mathematical arguments during the second data collection. S6 chose only to focus on two decimals in Q1, and in the first interview, he explained this was to meet the students at their level. S3 used percentages to explain her reasoning in the first interview, which resulted in giving a wrong answer to the question. The number S6 calculated (0.01) as the only number between 0.99 and 1, which was not a number between 0.99 and 1. This may be a case of confusing numbers and measurement as demonstrated in Yopp et al.'s (2011) study of in-service teachers. S6 held on to this explanation in I1, but in the third semester, he claimed that there were infinitely many numbers between any two different numbers if all numbers were to be considered. He answered using contradictory general and specific arguments. However, he was consistent in his opinion that there are no numbers between 0.99… and 1. S2 also had a contradictory concept image about 0.99… and 1, but he was consistent in his position that 0.99… and 1 were not equal. He stated that there were numbers between 0.99… and 1, and also that there were not. Both S2 and S6 were consistent with their answers on items 10–12 in Q2. However, confused by infinity, neither S6 nor S2 could unravel their contradictions.

S1, S4 and S8 tried to find specific numbers between 0.99… and 1. Failing in this endeavour made them respond in three different ways, but with the same conclusion. S1 kept her conviction that there were numbers between 0.99… and 1 and that they were different numbers. S4 changed his mind and stated that there were no numbers between 0.99… and 1 but they were different, nevertheless. S8 stated that 0.99… and 1 were different numbers, where 0.99… was an incomplete number. S1 stayed true to her general (correct) position that there were infinitely many numbers between any two different numbers, whereas S4 kept true to his specific argument and overlooked his general (correct) standpoint.

Discussion

Pre-service- and in-service teachers have over the years been reported to struggle with infinity, the number zero, division, and concrete contexts when interpreting division by zero and the real number line (e.g., Crespo & Nicol, 2006; Juter, 2019; Lajoie & Mura, 1998). Challenges remain though, for example regarding instructional practices (e.g., Karakus, 2018), which include argumentation. The aim of the present study is to further understand pre-service teachers' general and specific arguments in relation to mathematical structures developing over time in their education. The main contributions to the research field concern their (1) relations between general and specific arguments, (2) use of incorrect mathematical structures, and (3) incorrect conclusions from correct arguments and vice versa.

Several of the pre-service teachers' general and specific arguments were contradictory due to interpretation difficulties, particularly at the first data collection. This was apparent in the variety of specific arguments used with other mathematical structures than was intended for division by zero. Incorrect interpretations of concrete examples were also causing incoherence, for example, when distributing objects in zero piles with a focus on the objects at hand. A shift of focus occurred at the second data collection for some participants, which allowed them to conclude that the division was impossible. Conclusions were made from comparing different cases of division or operational relations. The modes of regarding the items had in those cases developed from concrete to abstract, as defined by Coles and Sinclair (2019). Dimmel and Pandiscio (2020) worked with another type of focus shift, from the impossible division to the singularity in a concrete dynamical computer setting. This showed a shift towards relations between objects, that is, relations between lines with different slopes. Relational comparisons worked in both studies as arguments for crucial differences and similarities in specific cases to allow general conclusions.

Another aspect causing contradictions for the pre-service teachers was the infinite recitation of nines in 0.99… in relation to the general rule that different numbers are separated by other numbers on the real number line. Even though some tried, but failed, to find numbers between 0.99… and 1, they still believed that there are such numbers. General arguments were in these cases correct and specific arguments wrong, in contrast to the result in a study about limits where general arguments often were wrong and specific arguments correct (Juter, 2007). The question of limits and attainability was part of the problems in both studies. In the study about limits the problem for many students was to accept that the general definition does not state that limits are unattainable for functions. The limit process causing problems in the present study was the never-ending recitation of nines in 0.99…, and to see 0.99… as another representation of 1. Infinity is an abstract concept (Tall, 2001; Tall & Tirosh, 2001) and the pre-service teachers' inabilities to find a specific counterexample was not enough to make them change their specific arguments to match their general arguments.

Evoking conflicting parts of concept images is essential for learners to become aware of inconsistencies (Tall & Vinner, 1981). Such insights also involve recognition of the amount of work required to obtain coherence. Most of the pre-service teachers in the study were under the impression that they knew enough mathematics to teach (Table 1). They had hence no incentive to scrutinise their own content knowledge to remedy possible inconsistencies. Teachers and pre-service teachers have been shown to prefer physical or practically based explanations (Lajoie & Mura, 1998; Levenson et al., 2010). Understanding the relation between an explanation and the mathematical rule or phenomenon explained is crucial for providing appropriate explanations and includes seeing the general in the specific. Assessing generalisations from specific examples requires adequate content knowledge (Font & Contrera, 2008). This is something that several pre-service teachers struggled with in this study. Their propensity to provide physical examples for explanation seem to be stronger than their abilities to scrutinise the examples' usefulness.

Correct conclusions in the study were sometimes argued for with incorrect arguments based on a variety of mathematical structures, as in S8's responses to the division items. Inversely, some incorrect conclusions were based on correct arguments. The argument "two divided by zero is not possible" led to the erroneous conclusion that the division therefore equals zero, in some cases. The conclusion might be drawn from a context where division always results in a number. This type of argumentation is a possible reason for the conclusion that division of a number by zero is equal to zero, as found in Crespo and Nicol's (2006) and Juter's (2019) studies. The new context, where division sometimes is impossible, leads to an incorrect conclusion, which did not correspond to other divisions such as zero divided by two. The result is a contradiction where the specific examples *two divided by zero* and *zero divided by two* are both equal to zero. Another example of traces of correct arguments in a wrong answer was given by S6. He argued that there is a finite number of numbers between 0.99 and 1 by adding a condition, which was to only consider hundredths. The strategy to focus on specific sets of numbers and not the entire real number line is an example of simplifying complex situations by using alternative models as described by Fischbein (2001) and Tall (2001). S6, however, confused *numbers* with *difference between numbers* and answered incorrectly in the conditioned context as well.

Conclusions

Overall, the pre-service teachers' content knowledge developed in the sense that they investigated relations to a higher extent at the end of the study than in the beginning. Some problems remained at the end though. Concrete specific arguments used as generic examples were part of the problems experienced as the arguments often were unsuitable for the contexts. Some contradictions through the study came from arguments with unintended underlying mathematical structures. The weak correspondence between the mathematical structures in the arguments and the contexts complicated generalisations. General and specific arguments were consequently contradicting in several of those cases. The weakly founded arguments led to both true conclusions based on erroneous arguments and false conclusions based on correct arguments. The results show the importance of creating situations that enable pre-service teachers to discover the state of their content knowledge from a meta perspective.

Implications for Practice and Further Research

Mathematical structures and representations of concepts were misused in several of the pre-service teachers' arguments in this study. Teachers' incoherent mathematics content knowledge can hinder attempts to make the content easier for students and may give students wrong impressions of what can be considered as mathematically true. The pre-service teachers' incoherent concept images imply that benefits could be realised by evoking contradicting parts simultaneously in activities that focus on relational aspects to remedy inconsistencies. Concrete arguments for a hypothesis explicitly linked to general arguments for the same hypothesis can help resolve cognitive conflicts and hence create a stronger concept image (Tall & Vinner, 1981). Further investigations of the effects of such efforts in teacher education would be helpful for mathematics teacher education development. Another research area for further study, considering these results, is in-service teachers' arguments about the topics in this paper. The problematic aspects remaining at the end of the pre-service teachers' mathematics education combined with Yopp et al.'s (2011) findings—that in-service teachers' misunderstandings may develop further in their profession—can lead to mathematics misunderstandings being passed on to students.

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References

Bryman, A. (2016). Quantitative and qualitative research: Further reflections on their integration. In J. Brannen (Ed.), *Mixing methods: qualitative and quantitative research* (pp. 57–80). Routledge. https://doi.org/10.4324/9781315248813

Coles, A., & Sinclair, N. (2019). Re-thinking 'concrete to abstract' in mathematics education: Towards the use of symbolically structured environments*. Canadian Journal of Science. Mathematics and Technology Education, 19*, 465–480. https://doi.org/10.1007/s42330-019-00068-4

Crespo, S., & Nicol, C. (2006). Challenging preservice teachers' mathematical understanding: The case of division by zero. *School Science and Mathematics, 106*(2), 84–97. https://doi.org/10.1111/j.1949-8594.2006.tb18138.x

Dimmel, J. K., & Pandiscio, E. A. (2020). When it's on zero, the lines become parallel: Preservice elementary teachers' diagrammatic encounters with division by zero. *Journal of Mathematical Behaviour, 58*, 100760. https://doi.org/10.1016/j.jmathb.2020.100760

Fischbein, E. (2001). Tacit models and infinity. *Educational Studies in Mathematics, 48*(2), 309–329. https://doi.org/10.1023/A:1016088708705

Flick, U. (2018). Triangulation in data collection. In U. Flick (Ed.), *The SAGE handbook of qualitative data collection* (pp. 527–544). SAGE Publications.

Font, V., & Contreras, A. (2008). The problem of the particular and its relation to the general in mathematics education. *Educational Studies in Mathematics, 69*(1), 33–52. https://doi.org/10.1007/s10649-008-9123-7

Gobo, G. (2018). Upside down: Reinventing research design. In U. Flick (Ed.), *The SAGE handbook of qualitative data collection* (pp. 65–83). SAGE Publications.

Juter, K. (2007). Students' conceptions of limits, high achievers versus low achievers. *The Mathematics Enthusiast, 4*(1), 53–65. https://doi.org/10.54870/1551-3440.1058

Juter, K. (2019). University students' general and specific beliefs about infinity, division by zero and denseness of the number line. *Nordic Studies in Mathematics Education, 24*(2), 69–88.

Karakus, F. (2018) Investigation of pre-service teachers' pedagogical content knowledge related to division by zero. *International Journal for Mathematics Teaching and Learning, 19*(1), 90–111.

Katz, K. U., & Katz, M. G. (2010). When is .999... less than 1? *The Mathematics Enthusiast, 7*(1), 3–30. https://doi.org/10.48550/arXiv.1007.3018

Lajoie, C., & Mura. R. (1998). The danger of being overly attached to the concrete: The case of division by zero. *Nordic Studies in Mathematics Education, 6*(1), 7–21.

Levenson, E., Tsamir, P., & Tirosh, D. (2010). Mathematically based and practically based explanations in the elementary school: Teachers' preferences. *Journal of Mathematics Teacher Education, 13*(4), 345–369. https://doi.org/10.1007/s10857-010-9142-z

Mason, J., & Pimm, D. (1984). Generic examples seeing the general in the particular. *Educational Studies in Mathematics, 15*(3), 277–289. https://doi.org/10.1007/BF00312078

Moreira, P. C., & David, M. M. (2008). Academic mathematics and mathematical knowledge needed in school teaching practice: some conflicting elements. *Journal of Mathematics Teacher Education, 11*(1), 23–40. https://doi.org/10.1007/s10857-007-9057-5

Sbaragli, S. (2006). Primary School Teachers' beliefs and change of beliefs on Mathematical Infinity. *Mediterranean Journal for Research in Mathematics Education, 5*(2), 49–75.

Shulman, L. S. (1986). Those who understand knowledge growth in teaching. *American Educational Research Association, 15*(2), 4–14. https://doi.org/10.2307/1175860

Stohlmann, M., Cramer, K., Moore, T., & Maiorca, C. (2014). Changing pre-service elementary teachers' beliefs about mathematical knowledge. *Mathematics Teacher Education and Development, 16*(2), 4–24.

Swedish Research Council. (2017, June). *Good research practice.* https://www.vr.se/download/18.5639980c162791bbfe697882/1555334908942/Good-Research-Practice_VR_2017.pdf

Takker, S., & Subramaniam, K. (2019). Knowledge demands in teaching decimal numbers. *Journal of Mathematics Teacher Education, 22*(5), 1–24. https://doi.org/10.1007/s10857-017-9393-z

Tall, D. (2001). Natural and formal infinities. *Educational Studies in Mathematics, 48*(2), 199–238. https://doi.org/10.1023/A:1016000710038

Tall, D., & Tirosh, D. (2001). Infinity: The never-ending struggle. *Educational Studies in Mathematics, 48*(2), 129–136. https://doi.org/10.1023/A:1016019128773

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169. https://doi.org/10.1007/BF00305619

Thanheiser, E., Whitacre, I., & Roy, G. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on whole-number concepts and operations. *The Mathematics Enthusiast, 11*(2), 217–266. https://doi.org/10.54870/1551-3440.1303

Tsamir, P. (1999). The transition from comparison of finite to the comparison of infinite sets: Teaching prospective teachers. *Educational Studies in Mathematics, 38*(1–3), 209–234. https://doi.org/10.1023/A:1003514208428

Tsamir, P. (2001). When 'the same' is not perceived as such: The case of infinite sets. *Educational Studies in Mathematics, 48*(2), 289–307. https://doi.org/10.1023/A:1016034917992

Tsamir, P., & Sheffer, R. (2000). Concrete and formal arguments: The case of division by zero. *Mathematics Education Research Journal, 12*(2), 92–106. https://doi.org/10.1007/BF03217078

Tsamir, P., & Tirosh, D. (2002). Intuitive beliefs, formal definitions and undefined operations: Cases of division by zero. In G. C. Leder, E. Pehkonen & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education? Mathematics Education Library, Vol 31* (pp. 331–344). Springer.

Vamvakoussi, X., & Vosniadou, S. (2012). Bridging the gap between the dense and the discrete: The number line and the "rubber line" bridging analogy. *Mathematical Thinking and Learning, 14*(4), 265–284. https://doi.org/10.1080/10986065.2012.717378

Yopp, D. A., Burroughs, E. A., & Lindaman, B. J. (2011). Why it is important for in-service elementary mathematics teachers to understand the equality .999... = 1. *The Journal of Mathematical Behavior, 30*(4), 304–318. https://doi.org/10.1016/j.jmathb.2011.07.007

Appendix

The pre-service teachers' answers to item *A* in Q1 were translated and presented in the column labeled Q1 and their responses to items 4 to 6 in Q2 are presented in the column labeled Q2 in Table 2*. The answers to item *B* in Q1 were translated and presented in the column labeled Q1 and their responses to items 7 to 12 in Q2 are presented in the column labeled Q2 in Table 3*. The texts in column Q1 are formulations translated from Swedish and the texts in columns I1 and I2 are summaries of responses in both tables.

Table 2*

Pre-service teachers' responses in relation to two divided by zero

	Q1, $\frac{2}{0}$	I1, $\frac{2}{0}$	Q2, $\frac{2}{0}$	I2, $\frac{2}{0}$
S1	2 persons are to share zero crowns. How much does each person get?	Shifts numerator and denominator: Two persons dividing zero crowns, then there is nothing to divide. Claims to realize the shift after it is pointed out, does not change her point of view: You cannot divide something that does not exist.	A number. [wrote = 0] Not infinity. Can be calculated.	Can only be 0. Two apples divided on 0 persons means that nobody owns the apples, the apples are still there. Compares with $\frac{0}{2}$: 0 has no value more than 0. Hard since we do not know where to place anything. Cannot explain $\frac{2}{0}$.
S2	If you try to divide something by zero, it becomes zero.	Shifts numerator and denominator: If you have nothing you cannot divide it in two. Sees the shift himself and reveals another misconception: You have two of something but nobody to share it with and get to keep both yourself.	Not sure if it is a number. Not sure if equal to infinity. Can be calculated.	First thought of $\frac{0}{2}$. $\frac{2}{0}$ equals a number. You have 2 that is not to be divided by anything, it becomes 2. Has never met this, unsure. Realizes that $2 \neq 0 \cdot 2$, the division cannot be done. Then it is equal to 0. Surprised it cannot be calculated.
S3	Zero is not possible to divide. If you have zero, you have nothing. Then it is hard to divide.	Shifts numerator and denominator: You cannot divide zero. If you have nothing in your hand, then you have nothing to divide. A concrete example makes a difference: Two pens cannot be distributed to zero kids.	-	-
S4	Imagine that you have two stones. You are to divide (put) these stones in zero piles. How do you do that?	Confusion in the Q1 answer: It feels like they still have their two stones. Finds it hard and tries to explain: If you have two and take it zero times.	A number. Not infinity. Can be calculated.	$\frac{2}{0} = 0$. 0 is not nothing. Compares with $\frac{0}{2}$: 0 is nothing, contradict myself. Cannot divide 2 by 0, nothing to divide 2 in so it becomes 0. We still have the objects, cannot place in piles. It becomes nothing but 0 still means something.

S5	If you have two apples to divide by zero. How many pieces of apples do you have?	Agrees with answer in Q1: If you have 2 apples and divide with no one then you have 2 left. Does not know the answer after a discussion about patterns.	A number. Not sure if equal to infinity. Can be calculated.	Thinks $\frac{2}{0}$ = 2. Compares with $\frac{0}{2}$: You cannot divide 0, you have 0 things to divide in 2, nothing to divide. $\frac{2}{0}$ is as if you have 2 things and nowhere to put them so you keep them. Realizes that it is division by 1 and concludes that you cannot divide by 0.
S6	If you have two crowns, but nobody to share it with. How many crowns do you have then?	States Q1 answer is wrong: If you take 2 and divide with 0 you do not get 2. Explains his answer 0: If you have two ships and they both sink, how many ships sink? Both objects should disappear, so you almost subtract to get 0.	Not sure if it is a number. Not infinity. Not sure if it can be calculated.	Did not get an answer when I thought about it. Compares with multiplication 2·0=0. I am not sure if you even can divide 2 by 0. Keeps reasoning. You can do $\frac{2}{2}, \frac{2}{1}$ but not $\frac{2}{0}$. Sees no way it can be done.
S7	[No answer]	Did not know at the time for Q1, still uncertain: Division with 0 does not mean anything, it would be hard to explain.	-	-
S8	You cannot divide 0 my friend. Never. You cannot divide a number that has no value.	Claims to agree with answer in Q1: You can never divide anything with 0. Claims to have expressed the answer in Q1 wrongly. 10 divided by 0 is nothing since you do not divide with anybody. If you divide with yourself, you get the whole thing.	Not sure on all the questions.	Does not know what $\frac{2}{0}$ is. Thinks $\frac{2}{0}$ = 0. After comparing with 0/2: Does not think division by zero is possible. Nothing to divide by. You keep the cake for yourself.

Table 3*
Pre-service teachers' responses in relation to the real number line

	Q1, 0.99 and 1	I1, 0.99 and 1	Q2, 0.99 … and 1	I2, 0.99 … and 1
S1	Take out the ruler and show the relation of mm and cm.	There are more numbers than what is visible: Ruler with invisible numbers between the marked ones. Does not know the answer. You can have infinitely many decimals.	Infinitely many numbers between any two different numbers. Two different numbers cannot be placed tight. There are numbers between 0.99 … and 1 so they are not equal. Disagrees with 10 and 11	There are numbers between 0.99 … and 1. Tries 0.9954, changes to that there probably are not. Changes her mind again at q12. We do not know where it stops. The numbers are not equal, there are numbers between them.
S2	I would draw a number line […] and ask what number I get if I add a zero after the nine, 0.990. If they said the same number, I would draw another number line and ask which numbers the line represents	Confirms the answers in Q1. Does not answer the question.	Infinitely many numbers between any two different numbers. Two different numbers cannot be placed tight. There are numbers, and also no numbers, between 0.99 … and 1 so they are not equal. Disagrees with 10	You can always add a 9. Confused by the infinite number of 9s, maintains that 0.99 … and 1 are not equal. Sticks to his contradicting answers in Q2.
S3	[No answer]	Finds it hard to explain: 0.99 is the step before 1. Uses % to explain: 1 is 100% and it is everything. We have a whole at 100. 99 is the step before 100.	-	-

S4	Try to start by explaining thousands. Draw a number line between 0.990 and 1.000.	Claims to use decimals to accommodate the students: After 0.99 there is 00000. Does not answer the question.	Infinitely many numbers between any two different numbers. Two different numbers cannot be placed tight. There are numbers between 0.99 … and 1, so they are not equal. Disagrees with 10 and 11	Cannot find a number between 0.99 … and 1 (tries 0.998). There are no numbers between, but they are not equal. Changes his answers on Q2 to agree on 11 and disagree on 10 and 12.
S5	I explain that there are numbers after 0.99. 0.990000 so there are many 0.991, 0.992, 0.993 etc.	Agrees with answer in Q1, with decimals making it possible to have as many numbers as you want.	Infinitely many numbers between any two different numbers. Two different numbers cannot be placed tight. There are no numbers between 0.99 … and 1 but they are not equal. Disagrees with 12	You can add another 9 after the infinite number of nines. But if there are infinitely many and you want to come to the next number, it is 1. Maintains that there are no numbers between 0.99… and 1 but they are different.
S6	There is only one number since 1 - 0.99 → 0.01. That is a hundredth that remains.	Focuses on hundredths and thousands to accommodate the students: You have a whole number [1] and subtract 0.99 then you get a hundredth. Does not answer the question.	Infinitely many numbers between any two different numbers. Unsure if two different numbers can be placed tight. There are no numbers between 0.99 … and 1 so they are not equal but also equal. Disagrees with 12	Number of numbers between numbers depends on number types. If all types, then infinitely many. About q9 (Q1): You can always add a 9 on 0.99. It never ends. About 10 and 11: It became messy in my head when I thought about infinity, but there is no number between 0.99… and 1.
S7	[No answer]	States that he knows this: It can be infinitely many because it depends on how many decimals you use.	-	-
S8	There are millions of different combinations.	Agrees with answer in Q1.	Not sure on all the questions.	Unsure if there are different numbers without numbers between. Suggests that 0.1 and 0.00001 are between 0.99… and 1. Tries again, fails. 0.99… and 1 are different numbers. 1 is whole, 0.99… is incomplete.

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