Analysis of Pre-service Teachers' Written Mathematical Reasoning

Madelyn W. Colonnese
University of North Carolina-Charlotte

Amanda R. Casto
Southern Oregon University

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Mathematical writing is one way for primary students to communicate their mathematical thinking. Research in the field of writing has shown that to become an effective teacher of writing, preservice teachers must have experience engaging in the kinds of writing given to their students. The study reported in this paper explored how 27 preservice teachers enrolled in an elementary mathematics methods course explained their reasons for selecting strategies to solve equations that required making either addition or subtraction calculations. Element scoring was used to analyse the preservice teachers' reasons for selecting the strategy and the composition of their mathematical writing. Results indicated that the majority of preservice teachers included a mathematical reason for selecting their strategy, however; the reasons varied in their connection to the mathematics discipline. This study contributes to the existing literature on mathematical writing by providing insight into how preservice teachers explain their mathematical reasons for selecting a computation strategy; thereby, demonstrating the need for increased opportunities for preservice teachers to write.

Keywords · mathematics teacher education research · mathematical writing · mathematical reasoning · primary preservice teacher education · computation

The concepts taught in primary mathematics teacher education courses are typically familiar to preservice teachers (PSTs). Mathematics methods courses are designed to support PSTs in learning how to teach such concepts. One instructional method recommended by the National Council of Teachers of Mathematics (NCTM, 2000) is mathematical writing. It follows that PSTs need to learn how to engage elementary students in mathematical writing about various concepts and procedures (AMTE, 2017).

Writing in mathematics provides students an opportunity to individually document their thinking, "see" their ideas (Colonnese et al., 2018), reflect on and clarify their thinking (NCTM, 2000), engage in an active construction of knowledge (Cross, 2009), and increase their use of formal vocabulary (Cohen et al., 2015). Importantly, writing to learn efforts, including mathematical writing, have been found to have a significant effect on students' learning of mathematics and ability to communicate (Graham et al., 2020). While there are many benefits to writing in mathematics, PSTs often come into teacher preparation programs with limited experience with this mode of discourse from their primary and secondary education (McCarthy, 2008), and therefore may not know how to support their future students with mathematical writing.

Researchers in the field of writing instruction found that for teachers to become effective teachers of writing, they need opportunities to participate in the kinds of writing given to their students (Graves, 1983; Grossman et al., 2000; National Commission on Writing, 2003). Since mathematical writing is a genre of writing, PSTs would benefit from engaging in the kinds of writing tasks given to their students. Such experiences could help PSTs become effective teachers of mathematical writing.

Given that PSTs often lack experience with mathematical writing and elementary students need opportunities to explain their thinking through writing (NCTM, 2000, 2014), it is worthwhile to study how PSTs communicate their mathematical thinking through writing. Such insights can inform aspects for mathematics teacher educators to consider when supporting PSTs in developing the skills needed to be effective mathematics teachers. The purpose of the study reported in this paper was to contribute to this area of research by exploring how PSTs enrolled in a course focused on methods for teaching mathematics to primary students explained their reasons for selecting strategies to solve an addition and a subtraction equation through writing.

Conceptual Rationale

Primary students who explain and justify their mathematical thinking have opportunities to deepen their understanding of the mathematical concepts (NCTM, 2014). Engaging students in such processes also contributes to the development of students' computational fluency, a central focus in the primary grades in the United States (Kilpatrick et al., 2001; NCTM, 2000). Computational fluency is also one of the Proficiencies of the *Australian Curriculum: Mathematics* (2013). Computational fluency includes the knowledge of multiple strategies, the ability to select a strategy that works best for a given expression or equation, and the skills to accurately apply the strategy consistently and efficiently (Baroody et al., 2014; NCTM, 2014; Rittle-Johnson & Star, 2007), as well as being able to explain application of the strategy (NCTM, 2014). For example, primary students who can compute fluently may think to solve 78 + 22 as (78 + 2) + 20. In the example of (78 + 2) + 20, students explaining their approach have an opportunity to record their individual understanding of partitioning 22 into two tens and two ones and the use of the associative property. In addition, students who are asked to justify their approach, may consider why this was or was not an effective strategy for solving the equation.

Importantly for primary students solving expressions such as 78 + 22 and writing about the process and/or justifying the approach supports the development of metacognitive skills (Cross, 2009; Pugalee, 2001). Metacognition includes a student's self-awareness and self-regulation of their cognitive processes (Campione et al., 1989), which are essential skills for successful problem solving (Davidson, & Sternberg, 1998; Hoffman & Spatariu, 2008; Schoenfeld, 1987). The permanent and explicit nature of writing offers students the ability to continuously monitor their progress while solving and then reflect on and evaluate the efficacy of their strategy and accuracy of their solution (Graham et al., 2013). The active nature of writing allows students to explore new ideas (Applebee, 1984). The exploration of new ideas is important for developing and refining computation strategies and adapting previously learned strategies to solve novel mathematics expressions/equations.

Although writing can offer students many benefits for learning mathematics, interpretations of what that means vary. Bossé and Faulconer (2008) suggested that the ambiguity in the use of writing during mathematics has resulted from failing "to distinguish reading and writing **about** mathematics from reading and writing **in** mathematics" (p. 10, original emphasis). Another factor is that the research literature on writing during mathematics classes includes a variety of definitions, types, purposes, and connections to the mathematics discipline (Colonnese, 2020). To clearly define mathematical writing, we used the purpose put forth by the Elementary Mathematical Writing Task Force—writing to reason and communicate mathematically (Casa et al., 2016). The Task Force's purpose aligns with the study focus: exploring the PSTs' written communication and reasons for selecting a strategy to solve both an addition and subtraction equation.

Mathematical Reasoning

Reasoning is central to learning and doing mathematics (Battista, 2017; Ball & Bass, 2003) and students need opportunities to explain their reasoning in writing (NCTM, 2000, 2014; Kramarski & Mevarech, 2003). *Adding It Up*, a seminal document in the United States exploring how students learn mathematics, described mathematics proficiency as five interdependent components, noting adaptive reasoning as one of those components (Kilpatrick et al., 2001). Adaptive reasoning can be described as "the capacity for logical thought and for reflection on, explanation of, and justification of mathematical arguments" (Kilpatrick, 2001, p. 107). Similarly, curriculum documents such as the *Australian Curriculum: Mathematics* define mathematical reasoning as "logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying, generalising" (Australian Curriculum Assessment and

Reporting Authority, 2013, p. 2). Both descriptions of mathematical reasoning emphasise logical thought and explanation. For the study described herein, we defined mathematical reasoning as logically justifying and explaining one's mathematical actions.

Researchers have noted that primary and pre-service teachers' knowledge of mathematical reasoning is often limited, with confusion about what reasoning entails (e.g., Simon & Blume, 1996; Loong et al., 2013; Stylianides et al., 2007). For example, one primary teacher in Loong et al.'s (2013) study when asked to describe what reasoning means shared, "I don't know … a lot to do with the understanding behind the mathematical concepts" (p. 470). In this example, the teacher struggled to articulate what was understood as reasoning.

Further, researchers have found that undergraduate students often connect mathematical reasoning to a well understood algorithm and rarely ground their reasons in the appropriate mathematical concepts (Bergqvist et al., 2008; Lithner, 2000). Given that there is limited understanding about what mathematical reasoning entails, it is warranted to support PSTs in developing a deep understanding of mathematical reasoning. Writing is one way for PSTs to engage in and communicate their mathematical reasoning.

Written Mathematical Reasoning

Explaining mathematical reasoning through writing is often challenging for students (Evens & Houssart, 2004; Kostos & Shin, 2010; Tan & Garces-Bacsal, 2013). Evens and Houssart (2004) found that when eleven-year-old students were asked to explain their reasoning, the students appeared to understand the mathematics but were unable to provide sufficient explanations. Monoyiou et al. (2006) noted that fifth- and sixth-grade students provided no justification or attempted to justify their thinking with a series of numerical examples. Similarly, in a study of eighth-grade students' expository writing, Shield and Galbraith (1998) identified a lack of reasoning and justification for students' mathematical algorithms. Norton et al. (2009) also realised this challenge in a study of mathematical letter writing between high school students and preservice teachers. The researchers found that reasoning and proof took longer to develop in students' writing.

Because it is likely that students will struggle with writing their mathematical reasoning, multiple opportunities during mathematics class to write and receive feedback on their written mathematical reasoning is needed. Primary teachers understand the importance of including writing during mathematics class (Powell et al., 2021). However; because of curriculum demands and/or the time needed to plan and implement a writing activity (Fukawa-Connelly & Buck, 2010), classroom teachers report using a low frequency of mathematical writing tasks, with few assessing the writing (Kosko, 2016; Powell et al., 2021). One way to address this challenge is to prepare PSTs to engage their future students in writing to reason mathematically. Drawing on the recommendations from researchers in writing education (e.g., Graves, 1983; Grossman et al., 2000; National Commission on Writing, 2003), for PSTs to develop the skills needed to implement mathematical writing, they need opportunities to personally engage in writing to reason mathematically during their teacher preparation programs. A better understanding of the challenges PSTs face when communicating their mathematical reasoning through writing can inform mathematics teacher educators' decisions about the kinds of opportunities for mathematical writing that PSTs need.

Mathematical Writing in Teacher Preparation Programs

Researchers from writing education recommend that teachers of writing need to be writers themselves (McDonald et al., 2004; Colby & Stapleton, 2006; Blau, 1988). Teachers "must be grounded on their own personal experience as writers-as persons who know first-hand the struggles and satisfactions of the writer's task" (Blau, 1988, p. 31). Further, Frank (2003) found that teachers' awareness of their own writing processes enhanced the teachers' ability to model the writing process for students and to identify and teach important writing strategies.

Many practicing teachers, however, do not feel comfortable teaching writing, with some feeling as though they did not learn to write well themselves as students (Morgan, 2010). Within mathematics teacher preparation, studies have revealed that the majority of PSTs have not used writing in mathematics and are often unsure about the benefits of using writing during mathematics class (Kenney et al., 2014; McCarthy, 2008; Ward, 2005). To address the lack of experience with writing, the National Commission on Writing (2003) recommended that universities should require writing as a component of PST's education. In line with these recommendations, some mathematics teacher educators have used different strategies to integrate writing within mathematics methods and content courses. For example, Kenney et al. (2014) implemented a writing to learn mathematics activity and a reflection. Danielson (2010) had PSTs write a paper that had three components: "(1) learn to perform the algorithm; (2) work a variety of problems using the algorithm; and (3) analyse the algorithm using the themes of the course" (p. 378). McCarthy (2008) engaged PSTs in using a graphic organiser to rehearse writing in the college classroom and then PSTs used the same graphic organiser in the primary school classroom. Ward (2005) incorporated children's literature and writing within a primary mathematics methods course. Each of these writing activities offered PSTs an opportunity to engage in mathematical writing or to read the writing of students.

After engaging with writing in mathematics, PSTs responded positively to this instructional practice and reported that they were likely to use writing with their mathematics instruction (Kenney et al., 2014; McCarthy, 2008; Ward, 2005). Danielson (2010) and McCarthy (2008) found that writing also provided the PSTs an opportunity to think critically about mathematical concepts. Such findings support the likelihood that by providing PSTs with opportunities to write they will use writing in their future classrooms. However, as Kenney et al. (2014) note, further research is needed to examine the use of writing tasks in mathematics teacher preparation programs.

Because PSTs have responded positively to writing mathematically and effective teachers of writers engage in the kinds of writing given to their students, our goal in the undergraduate mathematics methods course was to provide authentic opportunities for our PSTs to engage in mathematical writing. We decided to specifically focus on having our PSTs engage in written mathematical reasoning since this is an area of difficulty for many primary students and teachers (Evens & Houssart, 2004; Kostos & Shin, 2010; Tan & Garces-Bacsal, 2013). The purpose of this study is to describe the process for preparing and engaging PSTs in written mathematical reasoning and explore how the PSTs communicated their mathematical reasons for selecting a strategy to solve an addition and subtraction equation. Specifically, the research questions that guided this study are:

1. *How do preservice teachers communicate their mathematical reasoning in writing?*
2. *What are the challenges preservice teachers face with communicating mathematical reasoning through writing?*

Findings from this study can help to inform aspects to consider when preparing, engaging, and teaching PSTs about mathematical writing.

Method

Context and Participants

The participants in this study were a convenience sample of 27 PSTs at a large southeastern university in the United States. To ensure participant confidentiality, data analysis did not begin until the conclusion of the semester and all names were removed prior to analysis. Institutional Review Board approval was received for the study. Twenty-four of the participants were white females and three of the participants were white males. The participants were second- and third-year undergraduate university students completing their first semester of education courses. The course where the study took place focused on pedagogies and content knowledge for teaching mathematics to students in the primary grades. The course covered mathematics topics such as counting and cardinality, addition and subtraction within 1,000, geometry, and measurement, with a central focus on eliciting and interpreting student thinking. During the course PSTs are introduced to different ways to elicit student thinking such as questioning strategies and the use of writing.

Introducing and Engaging PSTs in Mathematical Writing

To prepare the PSTs for mathematical writing, the PSTs engaged in various activities to develop their understanding of reasoning and ability to write. Mathematical reasoning was introduced in the first two weeks of the course and then discussed throughout the semester. Collectively, we developed a shared understanding of mathematical reasoning after reviewing different student work samples and discussing NCTM's vision for mathematics teaching. NCTM's vision for mathematics teaching emphasises a high-quality education for all students that promotes aspects such as collaboration, participation in discussions, and engagement in challenging math tasks (2014). Then in Week 4 of the course, PSTs analysed and identified tasks that would provide students an opportunity to reason mathematically.

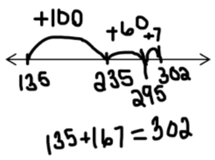
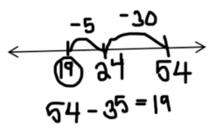
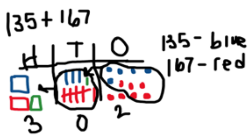
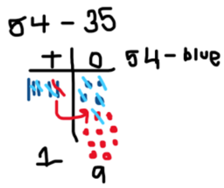
The mathematics content focus for the course when the study took place (Week 6) was computational fluency. During this portion of the course, the PSTs were introduced to the idea of Number Talks (Parrish, 2011) to help them learn about strategies for adding and subtracting. According to Parrish, a number talk is a brief discussion about the strategies used to solve a purposefully selected equation or expression. First, the PSTs watched a video of a teacher leading a number talk and were prompted to reflect on both the ways the teacher led the number talk and the students responded. PSTs were encouraged to analyse how students explained and justified their thinking. Next, we discussed how the responses by students demonstrated different kinds and levels of mathematical reasoning. Then, the PSTs planned their own number talk to implement in their field placement.

To develop the PSTs methods for teaching adding and subtracting multi-digit numbers, PSTs were asked to solve two equations, one with subtraction and the other with addition, using any strategy except the traditional algorithm (see Table 1 for an example of the traditional algorithm). The reason PSTs were asked not to solve using the traditional algorithm was because, in our experience, it is often the only strategy that PSTs think of when prompted to solve mathematical equations (AMTE, 2017). The goal of the activity was for the PSTs to use a variety of strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

The PSTs worked with partners to solve the equations on a whiteboard. Each group was asked to explain in words how they solved and why they selected that specific strategy. PSTs then walked around the classroom and discussed the various strategies their classmates used. As a class, we discussed the reasons for selecting a strategy, how to improve the justification for using a strategy, and the efficiency of the strategy selected for solving a particular equation. PSTs modified their responses with their partner based on the group discussion. Table 1 provides a description of the strategies the PSTs were introduced to during the course. The examples in the table represent one way to use the strategy. We recognise that there are different ways to apply these strategies to solve 135 + 167 and 54 – 35.

PSTs completed different tasks such as, explaining in writing why the expressions 22 – 9 and 23 – 10 had the same value. PSTs received feedback from the course instructor about the mathematical concepts and/or the composition of the writing. For example, one PST responded by stating, "The answer is the same, yes, 1 was added to 32 and 1 to 19 to still give the same answer 13." The course instructor provided feedback to the PST by asking questions like, "What if 1 was only added to the 32, would the expressions still be the same value? Why or why not?" The purpose of posing these questions was to help the PST understand ways to further expand the response and describe the justification for the claim made. Also, outside of class time, the PSTs read about different strategies used with subtraction and addition.

Table 1
Strategies for Addition and Subtraction

Strategy	Definition	Example	
		Addition	Subtraction
Partial Sums	Adds/subtracts the numbers of the same place value. Then adds/subtracts the total for each place value.	135 + 167 = 100 + 100 = 200 30 + 60 = 90 5 + 7 = 12 200 + 90 + 12 = 302	54 – 35 = 50 – 30 = 20 4 – 5 = -1 20 – 1 = 19
Making Friendly Numbers	Adds/subtract a certain amount to make computation more efficient.	135 + 5 = 140 167 + 3 = 170 140 + 170 = 310 310 – 8 = 302	35 + 5 = 40 54 – 40 = 14 14 + 5 = 19
Traditional or Standard Algorithm	Adding/subtracting starting from the ones place and regrouping when needed.	$^{1}1^{1}35$ + 167 302	^{4}~~5~~^{1}4 – 35 19
Number Line	Adds/subtracts using a number line.		
Concrete Model	Adds/subtracts using a representation such as base 10 blocks.		

After the PSTs received feedback about their response evaluating the expressions 22 – 9 and 23 – 10, the PSTs were asked to respond to the question in Figure 1 (used for analysis in this study). The PSTs responded to the question as part of a homework assignment and submitted responses through the online learning management system. Directions for the assignment were given in class, posted on the assignment, and PSTs were able to ask questions before responding. The PSTs' responses were not analysed as data until the conclusion of the semester.

Solve 135 + 167 = X using two different addition strategies.

Explain your mathematical reasoning for choosing each strategy to solve the equation.

Solve 54 – 35 = X using two different subtraction strategies.

Explain your mathematical reasoning for choosing each strategy to solve the equation.

Figure 1. The question prompt given to PSTs.

The equation in Figure 1 was created to align with the *Common Core State Standards* (CCSS) for Grade 2, to "add and subtract within 1,000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method" (2010, p. 19). The *Common Core State Standards* were used because this document was either adopted, adapted, or informed the mathematics standards in each state in the United States and the PSTs in the study were preparing to become teachers in the United States. We chose to incorporate the composition and decomposition of tens and hundreds in the equation because this is an important component of adding and subtracting in Grade 2 (Fuson & Beckmann, 2012). We used a writing feature "Explain your mathematical reasoning …" (Casa et al., 2019) in the question prompt to direct PSTs to include the mathematical reasons for selecting their strategy in their response. Lastly, the question prompt was reviewed by two researchers who are experts in mathematical writing. Both researchers were involved in leading a task force to define elementary mathematical writing, have collectively published several articles related to mathematical writing, and supported teachers with the implementation of mathematical writing. Feedback from the researchers was used to revise the prompt given to PSTs (Figure 1).

Data Analysis

To analyse the PSTs' writing, we reviewed other studies of students' mathematical writing to identify common approaches. Rubrics are one approach that has been used to assess mathematical writing (Cohen et al., 2015; Kasmer & Kim, 2012; Kostos & Shin, 2010; Lim & Pugalee, 2004). In developing a rubric to analyse the PSTs' responses, we first recognised that mathematical writing is multi-faceted including: content knowledge, general writing skills, writing skills and language that are specific to the mathematics discipline (Namkung et al., 2020). To capture the multi-faceted nature of writing, we analysed elements of the PSTs' mathematical reasoning and the composition of their mathematical writing. Element scoring provides a "detailed picture of the students' strengths and weaknesses, and instructional targets" (Namkung et al., 2020, p. 174). A detailed description of the PSTs' mathematical writing was necessary for answering our research questions because we were seeking to understand how PSTs communicated their mathematical reasons and the challenges they faced. We also selected this method because it has been shown to be reliable and valid in other studies (Hebert & Powell, 2016; Powell & Hebert, 2016).

We first analysed the PSTs' writing to identify common elements in the PSTs' written expressions of mathematical reasoning. We inductively coded the responses (Merriam, 2002) and identified three elements (described in Table 2): (1) connection drawn between the equation and strategy; (2) reason related only to the strategy; (3) related to future students. We also coded responses in a fourth category, no reasons present, such that the response did not include an element that described the PSTs' reasons for selecting the strategy. Table 2 provides the elements of mathematical reasoning, a definition, and an example from the data.

Table 2
Elements of Mathematical Reasoning for Strategy Selection

Element	Definition	Example (element is *italicised*)
Reasoning is connected to the equation and strategy	The response makes an explicit connection between the mathematical reasons for selecting a strategy and the equation.	"(135+167) +(-33+33) 102+200= 302 I used this strategy *because it is easier to add 200 and 102 than it is 135 and 167* because of the change in the place values on the number 167."
Reasoning is related only to the strategy	A response that makes a general connection between the mathematical reasons for selecting a strategy without referencing the equation	"135 + 167 = __; 100 +100 = 200; 90 + 12 = 302 I chose this strategy for the logical reasoning of *being able to decompose numbers into place values and being able to add the subtotals to produce the sum*."
Reasoning is related to future students	A response that appears to be driven by pedagogical decisions or assumptions for teaching the selected strategy	"The first strategy I used was I said 50 – 30 = 20 and then I did 5 – 4 = 1. Then I subtracted 20 –1=19. This gave me 19 as my final answer. This is a good strategy because *it helps students work with double digit subtraction that ends in 0 so it is a bit easier as well as a single digit subtraction problem that many students will have already memorised. Finally doing 20 – 1 = 19 is simpler, and students should not have too hard of a time with this*."
No reasons present	The response does not include a reason why the strategy was selected.	"One way I solved 54 – 35 is by decomposing it. I knew 30 + 5 is the same as 35. So, I took away the 30 from 54. Then I counted back the remaining 5 from 24 to get 19."

After identifying the three elements that described how the PSTs expressed their reasoning, we noticed that the reasons varied in sophistication. To understand such differences, we compared the elements we identified with other studies of students' mathematical reasoning. Evens and Houssart (2004) identified four categories: nothing on script, wrong or irrelevant, restatement, examples given/tested, some degree of justification. Lithner (2008) offered a conceptual framework to help characterise the different kinds of reasoning—imitative, algorithmic, and creative and the thinking process involved. Both studies analysed students' mathematical reasoning within the context of an argument or proof. We used the Lithner (2008) and Evens and Houssart (2004) studies to help differentiate the levels of reasoning while recognising that the PSTs in this study wrote explanations that were different from an argument (Casa et al., 2016). For example, Evens and Houssart (2004) note that some justification in an argument was more advanced than merely providing mathematical examples, as such we considered explanations where there was a connection made between the strategy and equation as more sophisticated than responses discussing related students.

In our analysis we also identified some responses that included multiple elements, for example a PST wrote a general reason why a strategy was selected and included why a future student might use the strategy. In this case the response was coded for both elements. Each of the PST's responses was analysed by both authors. The elements included in the responses were documented. Inconsistencies between the elements identified in the writing were discussed and consensus was reached. Additionally, we calculated the intercoder reliability (Miles & Huberman, 1994) as 92%.

Because mathematical writing is multi-faceted, we also analysed the PSTs' writing for elements unique to mathematical writing. We posited that by identifying the elements unique to mathematical writing we could more effectively realise the challenges PSTs face with regards to mathematical writing. To identify such elements, we reviewed recent standards initiatives including: NCTM's *Principles and Standards for School Mathematics* (2000), NCTM's *Principles to Action* (2014), the *Common Core State Standards* (CCSS) for Mathematics (National Governors Association, 2010) and AMTE's *Standards for*

Preparing Teachers of Mathematics (2017). We used the policy documents to identify the elements unique to writing because the documents reflect expectations for students' mathematical writing and thus, the skills PSTs need to be able to help their students develop.

We first identified mathematically precise language as an element of mathematical writing. The NCTM (2000) Communication Standard recommends that "by the end of high school years, students should be able to write well-constructed mathematical arguments using formal vocabulary" (p. 62). Additionally, the Common Core State Standard Mathematical Practice 6, suggests "mathematically proficient students try to communicate precisely to others" and "use clear definitions in discussion with others and in their own reasoning" (CCSS, 2010, para. 7). Further the Association of Mathematics Teacher Educators (AMTE) Standards recommend, "they [well-prepared beginning teachers of mathematics] use mathematical language with care and precision" (2017, p. 9).

The second element that we noted from the standards documents was that mathematical thinking should be communicated clearly, coherently, and follow a logical progression. We identified this aspect because the NCTM Communication Standard states that students' writing "should become increasingly sophisticated" (p. 60), and students should be able to "express themselves increasingly clearly and coherently" (p. 62). The Common Core Mathematical Standard for Practice 3 recommends that students "make conjectures and build a logical progression of statements to explore the truth of their conjectures" (National Governors Association, 2010).

Lastly, we identified the use of mathematical concepts including mathematical properties, meaningful use of symbols, and effective strategies as elements of written mathematical communication. The AMTE Standards recommend that "teachers can explain their mathematical thinking using grade-appropriate concepts, procedures, and language, including grade-appropriate definitions and interpretations for key mathematical concepts" (2017, p. 9). NCTM suggests that "explanations should become more mathematically rigorous and students should increasingly state in their supporting arguments the mathematical properties they used" (2000, p. 62).

We used the three elements identified from the standards documents: (1) mathematically precise language, (2) clearly, coherently, and follow a logical progression, and (3) use of mathematical concepts including mathematical properties, meaningful use of symbols, and effective strategies to inform the development of the framework used to code the PSTs' responses (Table 3). The definitions of the elements are specific to the question prompt in the study. We only coded the responses where a reason was present because the second research question addressed the challenges PSTs faced when communicating mathematical reasoning. Each response was coded by both authors. Inconsistencies were discussed and consensus was reached. We also calculated the intercoder reliability (Miles & Huberman, 1994) as 81%.

Table 3
Elements of Mathematical Writing

Element	Definition	Example (element is *italicised*)
Communicates precisely	PST uses more formal mathematical language conventional mathematics terminology accurately.	135 +167 200 + 90 + 12 = 302 Partial sums This strategy works because I *decomposed* the numbers into their place values: *hundreds, tens, and ones*. I added the subtotals together and that is how I got my answer, this works because it brings out the steps in a regular algorithm, making it easier to see how the addition works in an algorithm.
Communicates clearly, coherently, and logically	The ideas shared in the PST's response flow in a logical progression that is easy to understand.	*I added 5 to both numbers which made the problem 59 – 40. This is easy to look at and see that the answer is 19. It is much easier to work with a 40 and a 59 than it is to work with a 54 – 35.*
Use of mathematical properties	The PST includes a description of the properties of addition and place value.	This strategy works because I *decomposed the numbers into their place values; hundreds, tens, and ones*.

Findings

In order to answer our two research questions, we first describe the frequencies of the elements specific to mathematical reasoning (Table 4) and then those unique to mathematical writing (Table 5). We then provide examples of the PSTs' responses to further explain how the elements appeared in the PSTs' writing.

Elements Unique to Mathematical Reasoning

Thirty-eight of the 52 responses to the prompt involving addition and 22 of the 50 responses to the prompt involving subtraction included a mathematical reason (Table 4). The PSTs responses demonstrated differences in how they expressed their mathematical reasoning. The other responses collected did not include a mathematical reason.

Table 4
Frequency of Elements Related to Mathematical Reasoning

	Addition Equations (n = 52)		Subtraction Equations (n = 50)[a]	
Element	Strategy 1	Strategy 2	Strategy 1	Strategy 2
Reasoning is connected to the equation and strategy	10	8	7	6
Reasoning is related only to the strategy	9	8	2	5
Reasoning is related to future students	1	2[b]	1	1[b]
No reasons present	6	9	15	12
Total number of responses	26	26	25	25

[a]Some PSTs did not respond to the subtraction equation prompt.
[b]One PST addressed future students and the selected strategy as part of their reasoning. The response was coded in both categories.

Reasoning is connected to the equation and strategy

Thirteen responses for the equation involving subtraction and 18 responses for the equation involving addition were coded as making a connection between the strategy selected and the equation. The strategies used to solve the subtraction equation included (number of responses in parentheses): the traditional algorithm (1); friendly numbers (7); concrete model (3); number line (1); partial sums (1). For solving the addition equation, two strategies were used, partial sums (9) and friendly numbers (9). Nine of the PSTs who made a connection between the strategy and equation did so for at least one of their responses to the addition equation and for the subtraction equation.

We considered responses that drew a connection between the equation and strategy as the most advanced of the PST responses because they provided some mathematical justification for choosing the strategy to solve the equation. For example:

> 54 – 35 = X
>
> I decided to subtract 5 from each number because I wanted to make 35 30 to make it easier to subtract. Subtracting 5 from both sides gave me 49 – 30. This was easier to subtract than the original equation because I was able to use the standard algorithm without have to carry. 49 – 30 is the same thing as 54 – 35 because there is the same amount of numbers in between them.

The PST draws a connection to the equation by stating that they subtracted 5 from both "sides", meaning from the minuend and the subtrahend. The reason for selecting this strategy was to solve the equation without regrouping. Another PST shared:

> 135 + 167 = X
>
> Partial Sums: 100 + 100 = 200 30 + 60 = 90 5 + 7 = 12. Total: 302. I chose this strategy because it is a simple way to add two 3-digit numbers without having to carry over and keep the place, etc.

This PST described how adding within each place value also allowed them to add without regrouping, however; this PST did not acknowledge the need to regroup when adding the sub-totals together. It is also unclear as to what the PST meant by "keep the place." While both responses draw a connection between the strategy and equation, they also demonstrate the variation in how the PSTs explained their mathematical thinking about the mathematical concepts used to carry out their strategy.

Reasoning is related only to the strategy

A total of 24 responses included an element in the PSTs' response that described a general reason for selecting the strategy. Seventeen of the responses included the following strategies for addition: partial sums (9); traditional algorithm (3); concrete model (1); friendly numbers (4). Seven of the PSTs solved the subtraction equation using the (number of responses in parentheses): traditional algorithm (3); friendly numbers (1); partial sums (1); number line (2). The majority of PSTs' responses in this category generally stated that they found the strategy more efficient and that it provided them with a visual. Because the justification was grounded in the PSTs' personal perspective, we considered such responses as less advanced than those that drew a connection. For example:

> 135 + 167 = X
>
> Conventional algorithm by stacking vertical and adding down and carrying over the tens. By stacking the numbers, it is easier to visualise and see the numbers you are adding in the ones, tens, and hundreds place. This is a fast way and shows how to track and carry over the tens.

In this example, the PST explained why they personally find the conventional algorithm as more efficient rather than attending to the specific equation. The following PST used similar reasoning to describe why they selected the number line:

> 54 – 35 = X
>
> This strategy is breaking down the numbers to be whole but making it also a good visual aid. Using simpler numbers to better solve the problem, essentially solving it backwards.

One PST similarly shared that the strategy was more efficient, however; in using this strategy, a calculation error was made:

> 54 – 35 = X
>
> I broke the numbers down by place value and then reversed the numbers in the 1s place so there were no negative numbers. I then took 3 away from 20 to get 17. Breaking down the numbers into smaller parts allowed the problem to be less overwhelming. It allowed me to mentally solve the problem in a more efficient way than it would be to use the standard algorithm.

The PST changed the 3 tens to 3 ones when subtracting. Without drawing a connection back to the equation in the writing, the opportunity to notice the error in the solution was missed.

Reasoning is related to future students

Only two students based their mathematical reasoning on pedagogical decisions or assumptions about teaching. The following is an example:

> 135 + 167 = X
>
> The first strategy that I used was using partial sums method. The answer is 302. In this method, you would add the two ones in the hundreds place which is 200, the 6 and the 3 in the tens which is 90 and then the 7 and the 5 in the ones place which is 12. When all are added together, you get 302. This is a good strategy to use because it helps get a student's mind on thinking about the hundreds place, the tens place, and the ones place. It helps them also realise how the number is 302. It is important in addition to have a strong understanding of place value.

In the example, the PST acknowledged the importance of having a strong understanding of place value but does not explicitly state how using the partial sums strategy would contribute to this understanding and uses "two ones in the hundreds place" instead of a more precise description of "two one hundreds in the hundreds place." We considered such responses as less advanced than drawing a connection between the equation and strategy because the PST did not explain their mathematical reasoning for choosing the strategy. We did, however, recognise that the PST was considering how the strategy supported a student's understanding of place value.

No reasons present

Fifteen of the 52 (addition) and 27 of the 50 (subtraction) responses did not include an element describing the reason why the strategy was selected. For example:

> 135 + 167 = X
>
> I did expanded form for adding these numbers. First, I expanded each number. 100 + 30 + 5 + 100 + 60 + 7 Then I added the same place values together. 200 + 90 + 12 Then I added: 290 + 12 = 302.

In this example, the PST describes in words how they solved the equation but does not describe a reason for why this strategy was selected. Sometimes, a PST showed an attempt at explaining their reasoning with statements such as, "I know this works because…" or "I know I can…". For example:

> 54 – 35 = X
>
> Another way I did this was by breaking 35 into 30 + 5. I then did 54 − 30 to get 24. I then counted down 5 and arrived at 19. I knew I could do this because I rounded and subtracted that number and then just counted down what was left over for me to subtract.

This response lacks a claim as to *why* or *how* they know these methods work. As a result, such a response did not include any elements that shared the PSTs reasons why they selected the strategy.

Elements Unique to Mathematical Writing

Next, we analysed the elements unique to mathematical writing. We found that PSTs faced various challenges in clearly communicating their thinking. Table 5 details the frequency of the mathematical writing elements from the 60 PST's responses that included a mathematical reason.

Table 5
Frequency of the Elements Unique to Mathematical Writing

Element Unique to Mathematical Writing	Addition Reasoning (*n* = 37)	Subtraction Reasoning (*n* = 23)	Both	Neither
Communicates precisely	5	3	8	52
Communicates clearly, coherently, and logically	23	13	36	24
Use of mathematical properties	3	2	5	55
Total number of responses analysed			60	

Mathematically precise language

Eight PSTs' responses contained mathematically precise language. For example:

> 54 – 35 = X
>
> To solve this I used a number line, I started at 54 and counted backwards by 10s. 54--> 44--> 34--> 24 after counting back by 10's 3 times I counted back by one, 5 times. This strategy allowed me to visually see how I was solving the problem as well as show the relationship between the two numbers.

The PST describes their procedural steps precisely using the terms tens, ones, and a representation of their counting backwards by ten.

Fifty-two of the responses were identified as lacking mathematically precise language, including: using related terms, using incorrect terms, and missing terms. A related term was an informal word or phrase, but not the most precise one. For example, PSTs often used terms such as "round numbers" instead of "landmark numbers" or "borrow" instead of "regroup". Several responses, like the one below, contained the term "carry" instead of "regroup":

> 135 + 167 = X
>
> The stacking strategy is usually my go to when solving subtraction by hand, because it is a good visual aid. I chose this strategy because it is a simple way to add two 3-digit numbers without having to carry over and keep the place, etc.

Other responses lacked necessary words or phrases to precisely convey the PSTs' mathematical reasoning, for example:

> 54 – 35= X
>
> I had to borrow a one from the 5, making my 4 a 14. 14 − 5 is 9 and then I take the 4 – 3 and that creates 1, equaling 19.

The lack of using terms such as tens and ones when describing the numbers makes it difficult to determine which 5 the PST is referring to within their explanation. Overall, the data revealed that a majority of the PSTs in this study did not use mathematically precise terms to communicate their thinking.

Clear, coherent, and logical thinking

Most of the PSTs structured their responses in a clear, coherent, and logical manner. For example:

> 135 + 167 = X
>
> 1 hundred block + 3 tens rods + 5 small cubes + 1 hundred block + 6 tens rods + 7 small cubes
>
> 7 + 5 becomes 1 ten rod with 2 left over.
>
> There are now 10 tens rods which turn into 1 hundreds block.
>
> Now we have 3 hundreds blocks, 0 tens rods, and 2 small cubes (ones) = 302
>
> I used the drawing of manipulatives to add these two numbers because I knew which manipulatives represented hundreds, tens, and ones. I also knew how to regroup using the drawings.

Some of the PSTs' responses lacked clarity in the overall description of their strategy or the reasoning for selecting the strategy, for example:

> 54 – 35 = X
>
> I took 4 away from each to make the numbers easier to solve in the ones place. 50 − 31. and counted the difference from 31 to 50. This seemed to make it easier than 4 take away 5.

This response lacked enough information about why the PST first took away 4 from the minuend (54) and subtrahend (35), and then counted the difference. Further explanation of the PSTs' reasoning could have helped to illuminate why the decision was made to subtract 4 first.

Contains mathematical definitions or properties

Finally, we analysed the PSTs' responses to identify if they used definitions or properties. We found that only four responses contained definitions or properties. As in the example below, the PST provided a definition of the partial sums method:

> 135 + 167 = X
>
> This strategy is called the partial sums method, where you add the digits by place value. I chose this method because it helps me to organise the numbers in my head. I feel this method is helpful because it breaks down the larger numbers in a step-by-step strategy.

The inclusion of a definition helped to determine how the PST understood the strategy and also helped to describe why this strategy would make it easier for the PST to organise the numbers "in my head." Although the question did not specifically prompt for PSTs to include a mathematical definition or property, both the AMTE (2017) and NCTM (2000) Standards documents recommend that the use of mathematical definitions or properties elevates the mathematical rigor of an explanation.

Discussion

In this study, we explored how PSTs communicated their mathematical reasoning in writing and the challenges they faced. We examined the PSTs' written reasons, the use of precise mathematical vocabulary, the structure of the explanation, and the use of mathematical properties and definitions. The discussion that follows is organised by the two research questions.

Research Question 1: How do preservice teachers communicate their mathematical reasoning in writing?

Overall, the findings showed that a majority of PSTs included a reason for selecting their strategy in their responses. Because the task in this study called for the use of a strategy, it is reasonable that the PSTs drew on the algorithm when writing their mathematical reasons. This finding is consistent with other studies of students' mathematical reasoning (e.g., Bergqvist et al., 2008; Lithner, 2008). However; as noticed during the data analysis similar to Evens and Houssart (2004), the kinds of reasons varied across the responses.

We found the responses shared by the PSTs who relied on a strategy that they personally viewed as most efficient may or may not have been efficient for the given equation. For example, several PSTs selected the traditional or standard algorithm to solve the addition equation and explained that it was quick to use. While the traditional/standard algorithm produced an accurate solution, it may not be the most efficient strategy for computationally fluent students because they could use the relationship between the numbers and use a strategy reflective of the relationship (Russell, 2000). The finding that PSTs used a strategy regardless of the numbers, is similar to other studies of students' computational fluency (e.g., Boaler et al., 2015). This suggests that the PSTs may have memorised strategies for solving mathematical equations early on in their schooling rather than interacting with numbers flexibly. Mathematical writing tasks, like this one, can help teacher educators uncover PSTs' internalised beliefs about the efficiency of certain algorithms and more effectively support PSTs in critically challenging the use of certain rote learning procedures.

Mathematical writing also offers opportunities to engage in the metacognitive process (Pugalee, 2001), an important skill for computational fluency. The lack of connection between the PSTs' reasons for choosing a solution method and the numbers in the calculation suggests that the PSTs in this study may not have used writing to think critically about the strategy selected. This leaves questions about the PSTs preparedness to help their future students self-regulate when determining the most effective strategy. Helping PSTs become aware of their process when writing can enhance their ability to model and teach important writing skills (Frank, 2003). As evidenced by the findings in this study, PSTs may need opportunities to reflect on their mathematical writing to develop their metacognitive skills.

Similar to other analyses of primary and secondary students' mathematical arguments where students tend to struggle to explain their mathematical reasoning (e.g., Shield & Galbraith, 1998; Hoyles & Küchemann, 2002; Evens & Houssart, 2004), fifteen of the 52 addition and 27 of the 50 subtraction responses did not include a description of their mathematical reasoning. Because the PSTs were familiar with the mathematics, this suggests that the challenge for the PSTs may be the lack of exposure to mathematical reasoning and proof writing in their prior school experiences (Stylianides et al., 2007) or confusion about what reasoning entails (Loong et al., 2013). The lack of a mathematical reason in responses in this study may also indicate that PSTs could likewise struggle to communicate mathematically-situated reasons to future students, therefore reinforcing the use of procedures in a purely imitative manner without connection to underlying meaning (Lithner, 2008).

In general, the PSTs' writing in this study reveals that there is a need for additional opportunities to write mathematically. Students' ability to communicate effectively through writing typically improves quickly when engaged in several writing activities (Crespo, 2003; Norton et al., 2009) however; reasoning and proof take longer to develop (Norton et al. 2009). Engaging PSTs in reading exemplar responses and critiquing their own and their peers' writing and reasoning may further help to improve their ability to reason mathematically through writing.

Research Question 2: What are the challenges preservice teachers face with communicating mathematical reasoning through writing?

In analysing the PSTs' responses for elements unique to mathematical writing, the majority of responses were well-organised but lacked mathematically precise language. The PSTs often used informal or related terms to convey their ideas but not those accepted by the mathematical community as mathematically precise. Similarly, Seaman and Szydlik (2007) found that the primary PSTs in their study were mathematically unsophisticated, lacking enculturation into the mathematics community, and did not attend to the given mathematical language when solving tasks. Our findings add to the current understanding by identifying the common terms PSTs struggled with when explaining computational strategies, such as carrying and borrowing. Teaching PSTs to use terms like minuend and subtrahend can help them realise ways to more precisely describe a subtraction computation and can improve the quality of mathematical language their future students learn.

Thompson and Rubenstien (2000) noted that the knowledge of mathematics vocabulary is necessary for mathematics achievement and identify writing as one approach that can support the development of precise mathematical language. Further, Cohen et al. (2015) found significant differences between the use of formal vocabulary in second grade students who were engaged in a curriculum that emphasised written and oral communication than those who were not. An area for future research would be to compare PST written responses across a semester to explore if the use of precise mathematical language changes.

In addition to the lack of precise mathematical language, a description of a mathematical property or a definition were only identified in four responses. It is important to note that in those responses the PSTs correctly and appropriately applied the mathematical property or described the definition. Such findings suggests that few of the PSTs had a strong understanding of the mathematical concepts. In most cases, the PSTs did not use a mathematical property. Ding et al. (2013) found that the textbooks used by many PSTs did not provide conceptual support, which suggests resources used by PSTs may not always give them the opportunity to understand such concepts deeply.

This study demonstrates the need to investigate further PSTs' understanding of the mathematical properties related to addition and subtraction. Such research is warranted because the "mathematical foundations for understanding computational procedures for addition and subtraction of whole numbers are the properties of addition and place value" (Caldwell et al., 2011, p. 28). Additionally, PSTs need to develop skills to assess student thinking and design instruction that is responsive to student needs. Having a deep understanding of mathematical properties can help prepare PSTs to analyse student work and develop lessons that effectively connect student understanding to the related mathematics concepts (AMTE, 2017).

Limitations

Limitations within the study were noted. First, the PSTs responded to the writing task using the online learning management system. Typing the response can limit the kinds of representations that the PSTs used because it typically takes more time to upload a drawing. It may be helpful to create the task to encourage the use of paper and pencil or virtual manipulatives so that PSTs can use a variety of tools and representations to accurately convey their thinking.

Another limitation was that the audience for the PSTs writing was the course instructor. The audience impacts how one composes their mathematical writing (Casa et al., 2016). As a result, the PSTs may have responded in a certain way knowing that their instructor was reading and scoring the response. For example, knowing that their instructor would likely be familiar with the strategies they described in their writing even without the use of precise mathematical language, PSTs might have opted to explain their reasoning in more simplified terms. Further research should be conducted with varying audiences for the writing such as using pen pals (e.g., Phillips & Crespo, 1996) to see if there are changes in the ways that PSTs compose their mathematical writing.

Implications for Practice

This study offers insight to how a group of PSTs described their mathematical reasons for selecting a strategy to solve an equation. Writing tasks, like the ones used in this study, can help teacher educators identify PSTs who may need additional support with mathematical reasons and elements unique to mathematical writing. Additionally, writing tasks can offer opportunities for PSTs to further explore the mathematics (Danielson, 2010; McCarthy, 2008) through the application of mathematical reasoning and mathematically precise language. Writing tasks can provide the opportunity for teacher educators to give individualised feedback to their PSTs. Some areas where PSTs may need additional support when developing practices for mathematical writing include: 1) the precise mathematical language needed for the task; 2) the appropriate mathematical reasoning; and 3) the mathematical properties relevant for the task. Classroom activities to help provide such support may include more opportunities for PSTs to write in mathematics methods courses, analyse the writing of K–6 students, and critically analyse their own and their classmates' writing.

Conclusion

The purpose of the study reported in this paper was to explore how PSTs explained their mathematical reasons for selecting strategies to solve equations that required making either addition or subtraction calculations. Although varying in complexity, the majority of the PSTs included a reason for selecting a strategy for solving. The reasons given, varied in their connection to the equation solved. It also showed that few PSTs included reference to mathematical properties or precise mathematical language in their responses. Further research is needed to understand why PSTs may not have included these aspects in their responses. Additionally, studies as to how PSTs engage in non-routine problems where they need to engage in other kinds of reasoning are warranted as this may impact the elements in their written responses. Such work may also broaden PSTs understanding as to what mathematical reasoning entails, which is important to improving mathematics instruction.

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References

Applebee, A. N. (1984). Writing and reasoning. *Review of Educational Research, 54*(4), 577–596. https://doi.org/10.3102/00346543054004577

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics.* www.amte.net/standards

Australian Curriculum, Assessment and Reporting Authority. (2013). *Australian Curriculum: Mathematics, Foundation–10* (Version 8.4). https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 27–44). National Council of Teachers of Mathematics.

Battista, M. T. (2017). *Reasoning and sense making in the mathematics classroom: Grades 3-5*. National Council of Teachers of Mathematics.

Blau, S. (1988). Teacher development and the revolution in teaching. *The English Journal*, *77*(4), 30–35. https://doi.org/10.2307/819300

Bergqvist, T., Lithner, J., & Sumpter, L. (2008). Upper secondary students' task reasoning. *International Journal of Mathematical Education in Science and Technology*, *39*(1), 1–12. https://doi.org/10.1080/00207390701464675

Boaler, J., Williams, C., & Confer, A. (2015). Fluency without fear: Research evidence on the best ways to learn math facts. *Reflections*, *40*(2), 7–12.

Bossé, M. J., & Faulconer, J. (2008). Learning and assessing mathematics through reading and writing. *School Science and Mathematics, 108*(1), 8–19. https://doi.org/10.1111/j.1949-8594.2008.tb17935.x

Caldwell, J. H., Karp, K., & Bay-Williams, J. M. (2011). *Developing essential understanding of addition and subtraction for teaching mathematics in Pre-K–Grade 2*. National Council of Teachers of Mathematics.

Campione, J. C., Brown, A. L., & Connell, M. L. (1989). Metacognition: On the importance of understanding what you are doing. In R. I. Charles & E. A. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 93–114). Lawrence Erlbaum; The National Council of Teachers of Mathematics.

Casa, T. M., Firmender, J. M., Cahill, J., Cardetti, F., Choppin, J. M., Cohen, J., & Zawodniak, R. (2016). *Types of and purposes for primary mathematical writing: Task force recommendations.* http://mathwriting.education.uconn.edu

Casa, T. M., MacSwan, J. R., LaMonica, K. E., Colonnese, M. W., & Firmender, J. M. (2019). An analysis of the amount and characteristics of writing prompts in Grade 3 mathematics student books. *School Science and Mathematics, 119*(4), 176–189. https://doi.org/10.1111/ssm.12333

Cohen, J. A., Casa, T. M., Miller, H. C., & Firmender, J. M. (2015). Characteristics of second graders' mathematical writing. *School Science and Mathematics, 115*(7), 344–355. https://doi.org/10.1111/ssm.12138

Colby, S. A., & Stapleton, J. N. (2006). Preservice teachers teach writing: Implications for teacher educators. *Literacy Research and Instruction*, *45*(4), 353–376. https://doi.org/10.1080/19388070609558455

Colonnese, M. W. (2020). The development of instructional guidelines for elementary mathematical writing. *School Science and Mathematics*, *120*(3), 129–143. https://doi.org/10.1111/ssm.12391Colonnese, M. W., Amspaugh, C. M., LeMay, S., Evans, K., & Fields, K. (2018). Writing in the disciplines: How math fits into the equation. *The Reading Teacher*, *72*(3), 379–387. https://doi.org/10.1002/trtr.1733

Crespo, S. (2003). Learning to pose mathematical problems: Exploring changes in preservice teachers' practices. *Educational Studies in Mathematics*, *52*, 243–70.

Cross, D. I. (2009). Creating optimal mathematics learning environments: Combining argumentation and writing to enhance achievement. *International Journal of Science and Mathematics Education*, *7*, 905–930. https://doi.org/10.1007/s10763-008-9144-9

Danielson, C. (2010). Writing papers in math class: A tool for encouraging mathematical exploration by preservice primary teachers. *School Science and Mathematics, 110*(8), 374–381. https://doi.org/10.1111/j.1949-8594.2010.00049.x

Davidson, J. E., & Sternberg, R. J. (1998). Smart problem solving: How metacognition helps. In D. J. Hacker, J. Dunlosky, & A. C. Graesser (Eds.), *Metacognition in educational theory and practice* (pp. 47–68). Lawrence Erlbaum.

Ding, M., Li, X., & Capraro, M. M. (2013). Preservice elementary teachers' knowledge for teaching the associative property of multiplication: A preliminary analysis. *The Journal of Mathematical Behavior*, *32*(1), 36–52. https://doi.org/10.1016/j.jmathb.2012.09.002

Evens, H., & Houssart, J. (2004). Categorizing pupils' written answers to a mathematics test question: "I know but I can't explain." *Educational Research*, *46*(3), 269–282. https://doi.org/10.1080/0013188042000277331

Frank, C. R. (2003). Mapping our stories: Teachers' reflections on themselves as writers. *Language Arts*, *80*(3), 185–195.

Fukawa-Connelly, T., & Buck, S. (2010). Using portfolio assignments to assess students' mathematical thinking. *The Mathematics Teacher*, *103*(9), 649–654. https://doi.org/10.5951/MT.103.9.0649

Fuson, K. C., & Beckmann, S. (2012). Standard algorithms in the common core state standards. *National Council of Supervisors of Mathematics*, *14*(2), 14–30.

Graham, S., Gillespie, A., & McKeown, D. (2013). Writing: Importance, development, and instruction. *Reading and Writing*, *26*(1), 1–15. https://doi.org/10.1007/s11145-012-9395-2

Graham, S., Kiuhara, S. A., & MacKay, M. (2020). The effects of writing on learning in science, social studies, and mathematics: A meta-analysis. *Review of Educational Research, 90*(2), 179–226. https://doi.org/10.3102/0034654320914744

Graves, D. H. (1983). *Writing: Teachers and children at work*. Heinemann Educational Books.

Grossman, P. L., Valencia, S. W., Evans, K., Thompson, C., Martin, S., & Place, N. (2000). Transitions into teaching: Learning to teach writing in teacher education and beyond. *Journal of Literacy Research*, *32*(4), 631–662. https://doi.org/10.1080/10862960009548098

Hebert, M. A., & Powell, S. R. (2016). Examining fourth-grade mathematics writing: Features of organization, mathematics vocabulary, and mathematical representations. *Reading and Writing*, *29*, 1511–1537. https://doi.org/10.1007/s11145-016-9649-5

Hoffman, B., & Spatariu, A. (2008). The influence of self-efficacy and metacognitive prompting on math problem-solving efficiency. *Contemporary Educational Psychology*, *33*(4), 875–893. https://doi.org/10.1016/j.cedpsych.2007.07.002

Hoyles, C., & Küchemann, D. (2002). Students' understandings of logical implication. *Educational Studies in Mathematics*, *51*, 193–223. https://doi.org/10.1023/A:1023629608614

Kasmer, L.A., Kim, O.K. (2012). The nature of student predictions and learning opportunities in middle school algebra. *Educational Studies in Mathematics, 79*, 175–191. https://doi.org/10.1007/s10649-011-9336-z

Kenney, R., Shoffner, M., & Norris, D. (2014). Reflecting on the use of writing to promote mathematical learning: An examination of preservice mathematics teachers' perspectives. *The Teacher Educator, 49*(1), 28–43. https://doi.org/10.1080/08878730.2013.848002

Kilpatrick, J. (2001). Understanding mathematical literacy: The contribution of research. *Educational Studies in Mathematics*, *47*(1), 101–116. https://doi.org/10.1023/A:1017973827514

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics* (Vol. 2101). National Research Council. https://doi.org/10.17226/9822

Kosko, K. W. (2016). Writing in mathematics: A survey of K–12 teachers' reported frequency in the classroom. *School Science and Mathematics, 116*(5), 276–285. https://doi.org/10.1111/ssm.12177

Kostos, K., & Shin, E. K. (2010). Using math journals to enhance second graders' communication of mathematical thinking. *Early Childhood Education Journal, 38*, 223–231. https://doi.org/10.1007/s10643-010-0390-4

Kramarski, B., & Mevarech, Z. R. (2003). Enhancing mathematical reasoning in the classroom: The effects of cooperative learning and metacognitive training. *American Educational Research Journal*, *40*(1), 281–310. https://doi.org/10.3102/00028312040001281

Lim, L., & Pugalee, D. K. (2004). Using journal writing to explore: "They communicate to learn mathematics and they learn to communicate mathematically." *Ontario Action Researcher*, *7*(2), 17–24. https://oar.nipissingu.ca/archive-Vol7No2-V722E.htm

Lithner, J. (2000). Mathematical reasoning in school tasks. *Educational Studies in Mathematics, 41*, 165–190. https://doi.org/10.1023/A:1003956417456

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, *67*(3), 255–276. https://doi.org/10.1007/s10649-007-9104-2

Loong, E. Y. K., Vale, C., Bragg, L. A., & Herbert, S. (2013). Primary school teachers' perceptions of mathematical reasoning. In V. Steinle, L. Ball & C. Bardini (Eds.), *Mathematics education: Yesterday, today and tomorrow.* Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia, Melbourne (pp. 466–473).

McCarthy, D. S. (2008). Communication in mathematics: Preparing preservice teachers to include writing in mathematics teaching and learning. *School Science and Mathematics, 108*(7), 334–340. https://doi.org/10.1111/j.1949-8594.2008.tb17846.x

McDonald, J. P., Buchanan, J., & Sterling, R. (2004). The national writing project: Scaling up and scaling down. In T. K. Glennan, S. J. Bodilly, J. Galegher, K. A. Kerr (Eds.), *Expanding the reach of education reforms: Perspectives from leaders in the scale-up of educational interventions* (pp. 81–106). RAND Corporation.

Merriam, S. B. (2002). *Qualitative research in practice: Examples for discussion and analysis*. Jossey-Bass.

Miles, M. B., & Huberman, A. M. (1994). A*n expanded sourcebook: Qualitative data analysis*. SAGE Publications.

Monoyiou, A., Xistouri, X., & Philippou, G. (2006). Primary students' reasoning in problem solving and teachers' evaluation of their arguments. 2006. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education, Prague* (Vol. 4, pp. 177–184.

Morgan, D. N. (2010). Preservice teachers as writers. *Literacy Research and Instruction*, *49*(4), 352–365.

Namkung, J. M., Hebert, M., Powell, S. R., Hoins, M., Bricko, N., & Torchia, M. (2020). Comparing and validating four methods for scoring mathematics writing. *Reading & Writing Quarterly, 36*(2), 157–175. https://doi.org/10.1080/10573569.2019.1700858

National Commission on Writing for America's Families, Schools, and Colleges. (2003). The neglected "R": The need for a writing revolution. http://www.collegeboard.com

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.*

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.*

National Governors Association Centre for Best Practices, Council of Chief State School Officers. (2010). *Common core state standards*.

Norton, A., Rutledge, Z., Hall, K., & Norton, R. (2009). Mathematical letter writing. *The Mathematics Teacher, 103*(5), 340–346. https://doi.org/10.5951/MT.103.5.0340

Parrish, S. D. (2011). Number talks build numerical reasoning. *Teaching Children Mathematics*, *18*(3), 198–206. https://doi.org/10.5951/teacchilmath.18.3.0198

Phillips, E., & Crespo, S. (1996). Developing written communication in mathematics through math penpal letters. *For the Learning of Mathematics*, *16*(1), 15–22. https://www.jstor.org/stable/40248193

Powell, S. R., & Hebert, M. A. (2016). Influence of writing ability and computation skill on mathematics writing. *The Elementary School Journal*, *117*(2), 310–335. https://doi.org/10.1086/688887

Powell, S. R., Hebert, M. A. & Hughes, E. M. (2021). How educators use mathematics writing in the classroom: A national survey of mathematics educators. *Reading and Writing, 34*, 417–447. https://doi.org/10.1007/s11145-020-10076-8

Pugalee, D. K. (2001). Writing, mathematics, and metacognition: Looking for connections Through students' work in mathematical problem solving. *School Science and Mathematics*, *101*(5), 236–245. https://doi.org/10.1111/j.1949-8594.2001.tb18026.x

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology*, *99*(3), 561–574. https://doi.org/10.1037/0022-0663.99.3.561

Russell, S. J. (2000). Principles and standards: Developing computational fluency with whole numbers. *Teaching Children Mathematics*, *7*(3), 154–158. https://doi.org/10.5951/TCM.7.3.0154

Schoenfeld, A. H. (1987). What's all the fuss about metacognition. In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189–215). Lawrence Erlbaum.

Seaman, C. E., & Szydlik, J. E. (2007). Mathematical sophistication among preservice elementary teachers. *Journal of Mathematics Teacher Education*, *10*(3), 167–182. https://doi.org/10.1007/s10857-007-9033-0

Shield, M., Galbraith, P. (1998). The analysis of student expository writing in mathematics. *Educational Studies in Mathematics, 36*(1), 29–52 (1998). https://doi.org/10.1023/A:1003109819256

Simon, M. A., & Blume, G. W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *The Journal of Mathematical Behavior*, *15*(1), 3–31.

Stylianides, G. J., Stylianides, A. J., & Philippou, G. N. (2007). Preservice teachers' knowledge of proof by mathematical induction. *Journal of Mathematics Teacher Education*, *10*(3), 145–166. https://doi.org/10.1007/s10857-007-9034-z

Tan, T., & Garces-Bacsal, R. M. (2013). The effect of journal writing on mathematics achievement among high-ability students in Singapore. *Gifted and Talented International*, *28*(1–2), 173–184. https://doi.org/10.1080/15332276.2013.11678412

Thompson, D. R., & Rubenstein, R. N. (2000). Learning mathematics vocabulary: Potential pitfalls and instructional strategies. *The Mathematics Teacher*, *93*(7), 568–574. https://doi.org/10.5951/MT.93.7.0568

Ward, R. A. (2005). Using children's literature to inspire K–8 preservice teachers' future mathematics pedagogy. *The Reading Teacher*, *59*(2), 132–143. https://doi.org/10.1598/RT.59.2.3

Author Contact Information

Madelyn W. Colonnese
madelyn.colonnese@uncc.edu
University of North Carolina-Charlotte
9201 University City Blvd.
Charlotte, NC 28223

Amanda R. Casto
castoa@sou.edu
Southern Oregon University
1250 Siskiyou Blvd.
Ashland, OR 97520