Learning ambitious teaching of multiplicative properties through a cycle of enactment and investigation

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This study explores components of ambitious teaching practices for teaching multiplicative properties that can give teachers opportunities to learn through a quick images activity in a cycle of enactment and investigation. Throughout the cycle, it is found that the participants have opportunities to learn the following components of ambitious teaching: 1. the mathematical ideas (strategies and concepts) involved, 2. mathematical work on representations, 3. predicting student responses, 4. mathematical language, and 5. setting and working towards a focused mathematical learning goal for the students. The findings point out the complexities of learning to teach elementary mathematics, such as the associative property of multiplication. The findings also highlight the opportunities these complexities provide for teacher learning when it comes to distinguishing between and understanding commutative and associative computations.

Keywords ambitious mathematics teaching · multiplicative properties · opportunities for learning · cycle of investigation and enactment

Introduction

In a cycle of enactment and investigation, in-service teachers (ISTs) worked together with their supervisor with the aim of developing their teaching practices. The associative property of multiplication was loosely decided to serve as the mathematical focus and the instructional activity was organised around a quick image (Figure 1). Towards the end of the cycle (see methods section), IST6 asked the following questions:

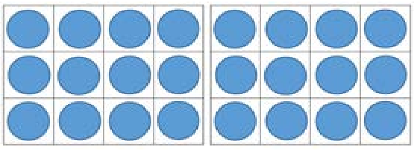


Figure 1: Quick image.

> Why is it called commutative [property for multiplication] when there are two [factors] and associative [property for multiplication] when there are three [factors]? What's the difference? Why couldn't we just use [the word] commutative, why is another word used there? It's just the same, isn't it? It's all about the order of the factors, or are they [commutative and associative properties of multiplication] two different things?

From the supervisors' and researchers' discussion after this cycle it was obvious that these questions came as a surprise since properties of multiplication – in the introduction to the cycle – were highlighted as possible mathematical focuses of attention when using quick images.

Providing opportunities for teacher learning and development is the core of professional development (PD). What ISTs learn from participating in Professional Development (PD) is thus one area of focus in educational research. In recent years, researchers have noted a shift in teacher education from teacher knowledge and abilities to teaching practice (e.g., Ball & Cohen, 1999; Forzani, 2014; Zeichner, 2012) or practice-based pedagogies (Hunter, Anthony, & Hunter, 2015). It has been argued that PD should aim to promote a close connection between the abilities of ISTs and the actual work of teaching (e.g., Grossman, Hammerness, & McDonald, 2009; Hiebert & Morris, 2012). This shift in focus from teachers to teaching runs parallel with efforts to conceptualise the work of teaching by identifying fundamental teaching practices. These teaching practices are sometimes referred to as *core practices* (e.g., Grossman et al., 2009; McDonald, Kazemi, & Kavanagh, 2013), *high-leverage practices* (e.g., Ball & Forzani, 2009, 2011) or *ambitious teaching practices* (e.g., Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010; Lampert et al., 2013). In the present study, we use the term *ambitious teaching practices* to refer to practices that enhance student learning of complex ideas and performances. How these practices are used in instructional dialogue is the key focus of ambitious teaching practices (Lampert et al., 2010).

Among the many promising initiatives aimed at developing pedagogies that promote opportunities for learning to enact ambitious teaching practices is the project *The Learning to Teach In, From, and Through Practice* (LTP). An essential idea in this project is to engage teachers through particular instructional activities in cycles of enactment and investigation (Lampert et al., 2010). These cycles involve observation, collective analysis, planning, rehearsals (Kazemi, Ghosseini, Cunard, & Turrou, 2016) and classroom enactment, including giving participants the opportunity to take short timeouts to make comments or ask questions (Fauskanger, 2019; Gibbons, Kazemi, Hintz, & Hartmann, 2017), followed by another round of collective analysis. Whereas the LTP project originally focused on supporting novice teachers in learning to enact ambitious mathematics teaching practices (e.g., Kazemi & Wæge, 2015; Lampert et al., 2010; Lampert et al., 2013), efforts have recently been made to adapt this pedagogy of ambitious mathematics teaching to the PD of ISTs (Gibbons, et al., 2017). Publications from these efforts conclude that more studies are needed in order to:

a) better understand how to support ISTs in learning to enact ambitious mathematics teaching practices, and
b) understand what ISTs can learn from participating in such PDs.

The present study elaborates on components of ambitious teaching practices that ISTs can learn, and this article reports from a study that is part of an ongoing school-based PD project, *Mastering Ambitious Mathematics teaching* (MAM), that implements these ideas in a Norwegian context. An overarching aim of this project is to investigate how a cycle of enactment and investigation (see methods section) might provide opportunities for ISTs to learn to enact ambitious mathematics teaching practices.

The aim of the shift to teaching practices (e.g., Zeichner, 2012) in teacher education and PD for ISTs has been to provide them with better opportunities to learn to carry out ambitious teaching practices that are fundamental to supporting students' learning of mathematics. As can be seen from the questions asked by IST6 presented above, teacher learning is not straightforward, even if IST6 was provided with opportunities to learn by participating in a cycle of enactment and investigation (e.g., Kazemi & Wæge, 2015). Whereas, for instance, Ghousseini and Herbst (2016) highlighted the importance of providing ISTs with opportunities to learn how to enact the complex work of teaching mathematics, Heyd-Metzuyanim, Tabach, and Nachlieli (2016) concluded that it is equally as important that teachers have opportunities to fail to enact. Bearing this in mind, this study addresses the following research question: What components of ambitious teaching practices do ISTs have opportunities for learning in a cycle of enactment and investigation? The instructional activity in focus involves a quick image and the mathematical focus of attention is multiplicative properties.

Theoretical Background

Learning to enact ambitious teaching practices through PD

Research on PD has been focused on what ISTs learn from participating (Kazemi & Hubbard, 2008). Researchers have tried to map out ISTs' learning by focusing on why and how change of practice is different among ISTs while engaging in PD (e.g., Borko, 2004; Kazemi & Hubbard, 2008). In their attempt to suggest new directions for the design and study of PD, Kazemi and Hubbard (2008) pointed out that improvement of teaching requires development of both knowledge and practice. More specifically, when student thinking is the focus of attention, Kazemi and Hubbard accentuated four crucial aspects that specify what ISTs must learn when responding to and elaborating on students' mathematical ideas. They must learn how to:

a) elicit and make sense of their students' mathematical thinking and reasoning,
b) choose and use mathematical ideas and representations that allow for productive and worthwhile mathematical learning trajectories,
c) orchestrate equitable classroom discussions and group work so that students productively engage with each other's ideas, and
d) monitor students' independent and group work to ensure that they develop conceptual and procedural understandings (Kazemi & Hubbard, 2008, p. 430).

These aspects are often included in descriptions of ambitious teaching practices. For instance, examples of ambitious practices proposed by Lampert et al. (2010) also have ambitious mathematical goals for the students' learning of mathematics and aim to elicit and respond to students' mathematical ideas in order to use students' knowledge and experiences as affordances and to build on their knowledge and experiences. Other examples of ambitious teaching practices are orienting students about each other's mathematical ideas and evaluating students' mathematical understanding. The mathematics teacher is challenged to carry out more than one of these ambitious practices at the same time and to continuously consider how and when to use them.

Kazemi and Hubbard (2008) proposed that learning to enact the teaching practices (a–d above) should be related to some routine instructional activities. Based on research that merges computational fluency and conceptual understanding, Lampert et al. (2010) have proposed instructional activities that target teaching and learning in the field of numbers and operations (see methods section). They claim that it is important that these instructional activities are structured with the potential to generate a variety of skills and knowledge, helping teachers to display what students have learnt and what they still need to learn. These researchers consider the way teacher routines are used in instructional dialogues to be a main component of ambitious mathematics teaching. At the same time, they suggest that one of the most challenging aspects of ambitious teaching practices is to maintain "a coherent mathematical learning agenda while encouraging student talk about mathematics" (Lampert et al., 2010, p. 9). The focus of attention in the present study is an activity relating to multiplicative properties. In the following, aspects of learning to teach multiplicative properties ambitiously will be presented.

Learning to teach multiplication ambitiously

Researchers have suggested that instead of developing routine expertise related to numbers and operations (Verschaffel, Greer, & De Corte, 2007), students should develop adaptive expertise defined by Hatano (2003) as "the ability to apply meaningfully learned procedures flexibly and creatively" (p. ix) A necessary step for developing adaptive multiplicative expertise is to understand the concept of multiplication and its arithmetic properties (Kilpatrick, Swafford & Findell, 2001). It may take several years to develop multiplicative understanding, and this requires much reconceptualisation of thinking on the part of the learner (Greer, 1992; Verschaffel et al., 2007). In accordance with these studies, Hurst and Hurrell (2014; 2016) pointed out that multiplicative thinking is one of the major ideas of mathematics that is fundamental to promoting the development of understanding mathematical concepts. Connections between different types of knowledge and between different aspects of a concept are seen as evidence of deep understanding (e.g., Hiebert & Carpenter, 1992; Lampert, 1986). In the following, we will present four models that represent multiplicative situations and focus on the learning of the arithmetic properties of multiplication.

Representing multiplication.

Models of multiplication are needed to illustrate different situations and unexpected results (Greer, 1992; Kilpatrick et al., 2001; Verschaffel et al., 2007). Such a need is prompted by different multiplicative situations, unexpected results (such as "multiplication makes smaller" when multiplying fractions) and strategies for calculation being underpinned by arithmetical properties. Four models are known to influence students' understanding of multiplication: equal groups, (rectangular) array, rectangular area and multiplicative comparison (e.g., Barmby, Harries, Higgins, & Suggate, 2009; Greer, 1992). These four models highlight different aspects of multiplication. Research has confirmed the impact of repeated addition as an intuitive model on multiplication, in which a number of groups of the same size are put together, that is, equal group situations (Fischbein, Deri, Nello, & Marino, 1985). According to these researchers, repeated addition does not view multiplication as commutative, i.e. 6×4 is interpreted as $4 + 4 + 4 + 4 + 4 + 4$ only. One factor (the number of equivalent collections) is taken as the operator; the other (the magnitude of each collection) as the operand. In Norwegian schools, multiplication is usually introduced by using equivalent groups. For example, 6×4 is normally interpreted as $4 + 4 + 4 + 4 + 4 + 4$ and 4×6 as $6 + 6 + 6 + 6$. Interpreting multiplication as adding the same number many times, i.e. repeated addition (Kilpatrick et al., 2001), leads to the equal group model. Even though the quick image in Figure 1 is an array, using it as an illustration, repeated addition can, for example, be illustrated by six equal groups of four dots (see Figure 4).

A second interpretation is the (rectangular) array interpretation of multiplication. If multiplication is thought of as a given number of rods of the same length end to end, the array model suits this interpretation (as the quick image in Figure 1). Looking at parts of the image only, the number of dots in the top row of the quick image in Figure 1 could be found by adding two fours (each seen as a rod) or by multiplying four by two (Figure 2b). Hurst and Hurrell (2014) highlight particularly multiplicative arrays as a powerful tool for representing multiplication because such arrays "have the potential to allow students to visualize commutativity, associativity and distributivity" (p. 9). In an on-going study with more than 400 primary-school students, Hurst and Hurrell (2016) have begun to investigate the opportunities afforded by multiplicative arrays in order to support children's understanding of multiplicative situations. A third model is the rectangular area interpretation of multiplication. This model, an extension of the array model, draws on continuous measures of length transformed to area by multiplication (Greer, 1992). In a fourth model, the multiplicative comparison model, two or more sets are compared. Finding the number of dots in a quick image twice as big as the image in Figure 1 asks for a comparison of two sets of dots, the one in Figure 1 and the one with twice as many dots.

Bearing the idea in mind that connections between different types of knowledge and between different aspects of a concept are seen as a sign of deep understanding (e.g., Hiebert & Carpenter, 1992), Larsson, Pettersson, and Andrews (2017) found that the students did not connect calculations to models for multiplication, although they showed a robust conceptualisation of multiplication as equal groups. According to these researchers, this "supported their utilisation of distributivity to multi-digits, but constrained their utilisation of commutativity" (Larsson et al., 2017, p. 1). The present study highlights the components related to the teaching of multiplication that give teachers opportunities to learn. This study also raises questions about instruction, emphasising that "introductions to multiplications might benefit from the use of models that would complement equal groups and support commutativity" (Larsson et al., 2017, p. 12).

Quick images are an instructional activity helpful for teachers' learning to enact ambitious teaching practices (Lampert et al., 2010). Quick images are typically presented through a configuration of dots as seen in the array in Figure 1, sometimes represented as chocolates in a box. A quick image could also be presented as an equal group model. These images are designed to help students to visualise numbers and form mental representations of a quantity by being invited to explain how they organised and subitised quantities in order to find the total amount of dots in the image. This activity was chosen according to Schumway's (2011) statement that the use of quick images builds on students' ability to compose and decompose a quantity, for instance by means of multiplicative properties.

Learning to teach the arithmetic properties of multiplication.
The arithmetic properties of multiplication relate to different models of multiplication (Schliemann, Araujo, Cassundé, Macedo, & Nicéas, 1998) and underpin multiplicative calculation strategies. Students' and ISTs'understandings of these properties are thus important. The commutative property of multiplication is an important property in that it reduces memorised number facts by almost half (Fuson, 2003). The development of an understanding of multiplicative commutativity is found to be dependent on instruction (Hurst & Hurrell, 2014; Schliemann et al., 1993).

The associative property of multiplication states that when multiplying three or more factors, it does not matter whether we first multiply the first two or the last two factors and then multiply their product with the last factor, as the final product will be the same (e.g., $2 \times (3 \times 4) = (2 \times 3) \times 4$). This property underpins strategies for multiplication. For instance, the associative property underpins the double-halving strategies, 16×25 can be written as $8 \times 2 \times 25$ by factorisation. The associative property can then be used to change the order of the calculations, $(8 \times 2) \times 25$, to $8 \times (2 \times 25) = 8 \times 50$. According to Larsson et al. (2017), little research has been conducted on students' understanding and teaching of associativity compared to commutativity (and distributivity). The distributive property of multiplication underpins written multiplication algorithms and focuses on partitioning a number (e.g., $12 \times 2 = 10 \times 2 + 2 \times 2$).

Combined, the three properties of multiplication allow freedom when multiplying. As an example, the factors 2, 3 and 4 invite multiplying in multiple orders (e.g., $2 \times (3 \times 4)$ and $3 \times (2 \times 4)$. Knowing the properties of multiplication (commutativity and associativity) leaves one product only. It is important for students to grasp the properties of multiplication (Hurst & Hurrell, 2014; Kilpatrick et al., 2001), but it is equally important to study components of ambitious practices for teaching multiplicative properties that give teachers opportunities to learn through participating in PD (Larsson et al., 2017).

Design, Methods and Analysis

The MAM project and its participants

The study presented here was conducted as part of the MAM project. In this project, a model and resources for school-based PD for in-service mathematics teachers (ISTs) in Norway were developed. The model builds on the approach to working with ISTs in a cycle of enactment and investigation (Lampert et al., 2013; McDonald et al., 2013), including instructional activities designed as "containers" for learning ambitious teaching practices. The structure of the activities (e.g., quick images, see Figure 1) offers scaffolding for eliciting and responding to student thinking and understanding. The ISTs learn to teach the activities through cycles of enactment and investigation. Each cycle consists of six stages:

1) The ISTs prepare for the cycle by reading supplied articles (e.g., about quick images) and by watching a video showing enactment of the cycle's activity. Some ISTs test the activity with their own students.
2) One of the supervisors leads a discussion on the literature as well as the video.
3) The groups of ISTs plan the given activity for given groups of students, supported by a supervisor.
4) In a rehearsal, one of the ISTs acts as the instructor. The supervisor and the other ISTs act as the students. During this rehearsal, all participants can ask for teacher timeouts (TTOs)
5) The same IST enacts the activity with a group of students. All participants can ask for TTOs. (For more about TTOs, see Fauskanger, (2019) and Gibbons, Kazemi, Hintz, and? (2017).
6) The enactment is analysed by each group of ISTs together with their supervisor. This analysis is followed by a similar analysis with all the participating ISTs, and preparation for the next cycle's activity.

The model and resources were piloted with groups of ISTs. This article reports from this pilot project.

School-based PD has many faces. One example is Lesson Study (e.g., Murata, 2011). When ISTs come together to teach in Lesson Study, one of the ISTs teaches according to a jointly developed detailed plan for the research lesson. The other ISTs observe and the discussion follows after the lesson. The cycles of enactment and investigation were developed in school-based PD ("Math Labs") (Gibbons, et al., 2017). The instruction is planned with fewer details than in Lesson Study, and known activities are used so the planning will take less time. As in Lesson Study, all ISTs are responsible for the teaching, but in "Math Labs" they are invited to contribute when the lesson is enacted, for instance by asking for TTOs.

The principals at ten schools selected ISTs they thought could subsequently serve as mentors for their colleagues so they could implement ambitious teaching practices in the entire school. Out of the 30 selected ISTs, 21 agreed to take part in the research in addition to being participants in the PD. They were divided into three groups, and two groups (14 ISTs) were randomly chosen to be part of the research (group 2 and group 3). Most of the participating ISTs taught grades 5–7 (i.e., students who are 11–13 years old). The age range for the teachers was 23 to 59 years, their teaching experience varied from one to 30 years and their formal education in mathematics/mathematics education varied between 15 ECTS (one year full time studies is 60 ECTS) and 180 ECTS. The present study explores what components of ambitious teaching practices the ISTs have the opportunity to learn independently of education, experience, and age.

In agreement with the principals, the project ran over four semesters, a half year each, with three four-hour cycles each semester held at one of the participating schools. Two of the cycles did not include all six stages. The school provided access to students during 'normal' class time and each group enacted the activity they planned with a group of 15–20 year-six students.

Methods for data collection

All components in the cycles of enactment and investigation were video-recorded and all the written material given to the ISTs was collected (e.g., lesson plans produced by the ISTs during the cycle). In addition, at least one of the researchers was present during each of the cycles. For the purpose of this article, the videos from one whole cycle (the fourth cycle) and from one group of ISTs, group 2, (seven ISTs and one supervisor) were analysed. Moreover, a discussion between supervisors and researchers after the cycle was included in the analysis, giving analysed video clips as presented in Table 1. After having watched all the video recorded cycles, we decided to delve deeper into one of the cycles in which there was an explicit discussion on multiplicative properties. Multiplicative thinking is one of the mathematical ideas that underpins fundamental features for students to learn in order to develop a relational understanding of mathematical concepts (Hurst & Hurrell, 2014).

Table 1. Analysed video material – an overview.

MAM cycle	Group	Discuss-ion (all groups)	Plan-ning (gr2)	Re-hearsal (gr2)	Enact-ment (gr2)	Discuss-ion (gr2)	Discuss-ion (all groups)	Discussion (super-visors and research-ers)
4	2	24:51*	59:21*	21:29*	26:46*	19:56*	29:42*	58:18*

*The numbers state the total length of the video clips.

The introduction of the activity was also included in the analyses. The instructional activity used in this session was a quick image (Figure 1). The overall aim of the activity was defined by the course instructors as to "build on different student strategies and discuss relations." As part of their preparation, each group had to formulate a more specific learning goal for the students (e.g., a property of multiplication) and plan the activity to attain the goal while building on students' thinking. All four groups discussed goals for the activity that related to the associative property of multiplication.

Analysis

The unit of analysis consists of the different stages of the cycle (see Table 1) with a particular focus on the chosen group and its planning, rehearsal, enactment and discussion after enactment.

When analysing the videos, we focused on the different components of ambitious teaching practices the participants had opportunities to learn in the cycle of enactment and investigation. Conventional content analysis was used. In conventional content analysis, researchers immerse themselves into the data material without predefined codes and categories so they can gain new insights (Hsieh & Shannon, 2005). This approach is seen as a flexible way of analysing rich data and a systematic approach to classifying and identifying themes or patterns (Hsieh & Shannon, 2005).

The two authors of this article coded the utterances independently, developed individual codes and grouped them into categories in an iterative process weaving back and forth between the empirical material and theories (Alvesson & Karreman, 2011). The focus of the analysis was to identify components of ambitious mathematics teaching practices the ISTs have opportunities for learning (Ghousseini & Herbst, 2016) through the quick images activity. In the analysis, ISTs also have opportunities to learn when they fail to enact the complex work of teaching mathematics (Heyd-Metzuyanim et al., 2016). After this coding process, the researchers worked together to determine the categories and corresponding codes. In this iterative process, we did not calculate any form of agreement rate, but we had a focused discussion on our codes and categories. Some minor adjustments to the codes were made during this process, but the listed categories remained the same with the following agreed-upon common categories:

- the mathematical learning goal for the students,
- the mathematical ideas (strategies and concepts) involved,
- the prediction of student responses,
- the mathematical language, and
- the mathematical work on representations.

The analysis was conducted in the original language and the representative utterances used to present the findings in this article were translated into English.

Findings

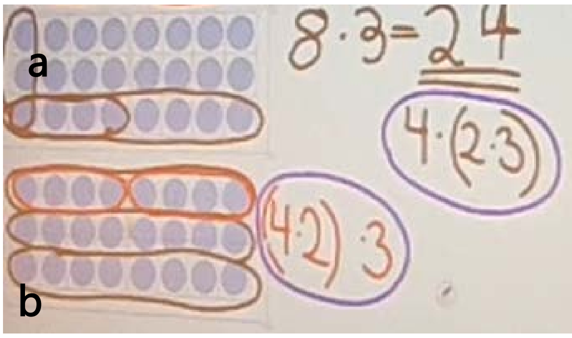
An introductory discussion (all groups)

In this session (Table 1, column three), the importance of formulating and working towards a mathematical learning goal for the students' learning when using quick images was highlighted by one of the supervisors. Moreover, possible learning goals were discussed, some general (e.g. communication, discussion, provoking curiosity) and some related directly to mathematics (the commutative, associative and distributive properties of multiplication and generalisation). It was pointed out that a quick image should be chosen according to the goal set for what the students were supposed to learn from the activity and that the commutative, associative and distributive properties of multiplication (mathematical ideas/strategies/concepts involved) might require different quick images (e.g., quick images as arrays or images constructed of equal groups of dots).

A concrete learning goal was not agreed upon during this introductory session. *Prediction* and challenges related to prediction were suggested by the supervisors, as well as by the ISTs, and were highlighted as an important aspect of the forthcoming planning phase of the cycle. Possible student responses were discussed, in addition to how to represent students' responses (*mathematical work on representations*) in the quick image and how to make symbolic representations (*mathematical language*).

Planning

The supervisor initiated the planning discussion by suggesting that the ISTs should predict student responses (predicting) before setting a goal for the students' learning. In this part of the planning phase (00:00–22:00) the supervisor went to the smartboard and wrote down mathematical ideas that were introduced by the ISTs (mathematical ideas involved) based on student responses from testing the quick image in their own classrooms. These mathematical ideas/strategies were then illustrated on a quick image (mathematical work on representations) before they discussed how to write the mathematical ideas by using symbols (mathematical language). The students' mathematical ideas functioned as possible predictions that might appear when testing this quick images activity in the enactment phase. Based on the ISTs' experience of their own classrooms, five different mathematical ideas/strategies were discussed. The first one was the following idea: "I see four times three, twice" (IST2). The supervisor represented this idea in the quick image and wrote 4 × 3 × 2 on the smartboard. We observed how the four categories (in italics above) were related and provided the ISTs with opportunities for learning components of ambitious teaching practices by paying attention to the importance of representing predicted mathematical ideas in the quick image, as well as expressing them in a mathematical language. This was further illustrated by two of the other mathematical ideas that were reported from the ISTs based on student responses from their classroom: 1) "I know that there are three in each row [column], then I counted eight in each row" (IST2) and 2) "I saw that there were eight across, then there were three eights" (IST6, cf. Figure 2ab which is from the enactment phase). Considering these ideas, the supervisor initiated a discussion of the commutative property (i.e., 8 × 3 = 3 × 8).



*As can be seen from this figure (and also Figures 3-5), the multiplication sign used is "·". This is the most common sign used in Norway. In the article, however, we have decided to use "×".

Figure 2. Two student strategies from the enactment phase.

In the second part of the planning phase (22:00–51:40) a component of ambitious teaching practices was also provided for ISTs' learning. This component was initiated by IST2 who was concerned about students seeing twelve as three times four by using parenthesis (3 × 4), followed by the utterance: "Is then three times four times two [(3 × 4) × 2] wrong? Is it right thing to write two first?". This was elaborated on by the example: (3 × 4) × 2 and 2 × (3 × 4) and the communicative property of multiplication was brought into the discussion. From this particular example, the supervisor wrote the general law of associativity on the smart-board: (a × b) × c and a × (b × c) and invited the ISTs to connect the general associative property with the particular example (3 × 4) × 2 in the quick image (Figure 1). From this part of the planning phase it seemed that some of the participants were mixing the commutative property of multiplication (i.e., (3 × 4) × 2 = 2 × (3 × 4)) with the associative property (i.e., (a × b) × c = a × (b × c)), indicating that these properties are not always easy to understand (Hurst & Hurrell, 2014; Larsson et al., 2017; Schliemann et al., 1998). IST7 responded and expressed: "Then b equals four [b = 4], c equals two [c = 2]. What is four times two here? I can't see that [in the image]." IST2 followed this by saying: "Four times two, three times" and the supervisor illustrated (4 × 2) × 3 in the quick image. These utterances show how the supervisor and the ISTs discussed *mathematical ideas involved* by representing the associative property of multiplication in a quick image, relating these examples to the general property of associativity. However, the utterances also illustrate that the participants seem to mix the associative and commutative properties of multiplication and they struggle in their *mathematical work on representations* (i.e. to represent 4 × 2 in the quick image). Participating in cycles of enactment and investigation thus provides opportunities to learn mathematical knowledge for teaching (Ghousseini, 2017).

A few minutes into the second phase (27:10–28:40), the supervisor challenged the ISTs to consider if they should extend the quick image: "Is it possible to extend [generalise] the quick image? Could we then think of how to extend [it]?" This idea was introduced while the group was discussing the examples reported on in the previous paragraph. Later in the discussion (40:25–42:20), IST2 followed up on this idea of generalisation: "One thing that would have been exciting to ask [is]: What if there had been one more row?" IST2 also stated that it is interesting to ask such questions because the students will then have to think about what would happen if they had more rows in the quick image.

In the second part of the planning phase (22:00–51:40) the supervisor challenged the ISTs to focus on possible *mathematical learning goals* for the students: "Has anyone thought about suggestions for possible learning goals we can use for the enactment? What mathematical ideas do we want to focus on [when using this quick image]?" Three more times during this second phase, the supervisor interrupted the ISTs' discussion to focus on a *mathematical goal for the lesson:* "We need to come up with a [learning goal]" and some minutes later: "I'm still not seeing a common [learning goal]. What is the mathematical idea we want to focus on?" The ISTs were concerned with different ways of seeing patterns and seemed to have difficulties in making the mathematical learning goal clear and focused. The fourth time the supervisor challenged the ISTs to consider a mathematical learning goal (about 45 minutes into the discussion), she herself suggested a goal from their previous discussion: "Yes, what mathematical idea should we focus on? Is it the associative property or is it more to develop a general expression for the total amount of chocolates in the boxes? [the dots in the quick image represent pieces of chocolate]." They agreed to focus on the associative property, and discussed and tried to predict how different suggestions from the students would help them to approach the associative property. For instance, "If 8 × 3 appears" (IST2), they wanted to challenge the students to use the factors 2, 3 and 4, and they discussed different combinations of these factors. In the third part of the planning phase (51:40–59:25) their focus was on practical teaching strategies and how to structure and teach the activity for the students.

To summarise, the analysis of the planning phase illustrated some components of ambitious teaching practices for teaching multiplication that were made available for ISTs to learn:

- the relation between mathematical, predicted ideas/strategies and the work on mathematical representations expressed in a mathematical language,
- the associative property of multiplication,
- the idea of generalising/extending the activity, and
- setting a focused mathematical learning goal for the students.

Rehearsal

The rehearsal started with a discussion initiated by the supervisor about how to introduce the activity on the smartboard, emphasising the importance of connecting different representations: from chocolates in boxes to a simpler representation by illustrating the chocolates as dots in quick images (Figure 1). IST6, who was going to teach the activity (hereafter instructor, where the ISTs took turns being the instructor), invited the other participants to act as students and propose ideas or strategies. Three such mathematical ideas appeared in the rehearsal. These mathematical ideas were illustrated in a quick image (mathematical work on representations) before the participants discussed how to write the mathematical ideas in a mathematical language. After having discussed the two first ideas (i.e., 2 × 12 and 4 × 6), the following input was introduced into the discussion by IST2: "I saw another pattern: eight in each row", and IST2 went to the board and showed three groups of eight dots (Figure 2b. The instructor wrote 3 × 8 on the board based on input from IST2 and continued with the following utterance: "In which way is it possible to see the eight here [points to the first row]." IST7 went to the board and illustrated two groups of four dots in the quick image (Figure 2b, first row), and the instructor wrote 3 × (2 × 4) based on the input from this colleague.

Based on the three discussed examples, the instructor stated: "We have three factors in each and then they [the students] can talk together about what they see." The participants agreed that it is "a long way from here to the introduction of letters [(a × b) × c = a × (b × c)]" (IST4). When rehearsing the lesson, the goal set for the activity was briefly mentioned towards the end of the session when the supervisor stated that these examples made a good structure and starting point for focusing on the associative property. The ISTs, however, decided to ask the students about "other ways to see the numbers" (IST5, e.g., 6 as 2 × 3) to have the three factors necessary for focusing on the associative property of multiplication.

From the analyses of the rehearsal phase it was clear that *mathematical ideas involved, mathematical work on representations* and *mathematical language* were the most prominent components of ambitious teaching practices made available for the participants to learn in the rehearsal phase.

Enactment

In the enactment of teaching, mathematical work on representations was visible when the activity was introduced by using an image of chocolates, as well as when the instructor illustrated the students' strategies by using dots (Figure 1). The mathematical ideas were visible when the students presented their strategies. The first student said that she saw two boxes and that in each box there were "four across and three down", which equals twelve and "then I took [multiplied] twelve by two." Guided by the student, the instructor represented this student's strategy in the quick image as three equal groups of four dots (Figure 3). It was, however, not clear whether the student actually meant three groups of four dots, or an area model for multiplication (e.g., Barmby et al., 2009; Greer, 1992). The student was invited to write her strategy on the board. She wrote "(4 × 3) × 2" (mathematical language) on the board (Figure 3).

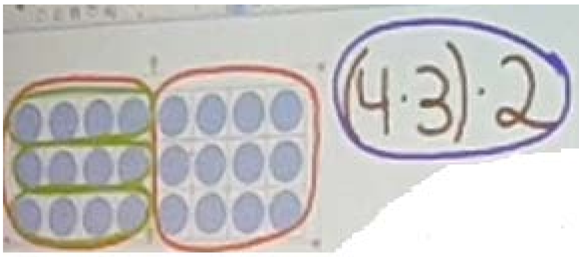


Figure 3. Two times three equal groups of four dots.

A second student presented her strategy as follows: “I saw a box with four across and three up, then there were two boxes and I took [multiplied] eight times three” parallel to representing the strategy in the quick image as illustrated in Figure 2a. When invited to write his strategy on the board, he wrote 8 × 3 = 24. This was followed by the instructor representing the strategy in a separate quick image as in Figure 2b. From this student’s representation in Figure 2a, it was evident that his model was an area model (e.g., Barmby et al., 2009; Greer, 1992). When the instructor represented this strategy in a second quick image (Figure 2b), it indicated that she did not recognise the area model, but preferred the equal groups model.

Pointing to the quick image (Figure 2b), the instructor asked the students to focus on the eight dots in the first row and to discuss how to “split eight into another multiplication task.” When the students did not respond to this question, the supervisor asked for a timeout (Fauskanger, 2019; Gibbons et al., 2017) to focus the students’ attention on the strategy already presented and asked how this student “knew that there were eight [dots] in the row he saw.” A student responded that there were four in each box, written as four times two. The instructor summed up this discussion by writing (4 × 2) × 3 on the board.

Approximately sixteen minutes into the enactment phase, a third student presented her strategy as seeing four six times in the quick image. The instructor circled four dots six times in the image (Figure 4).

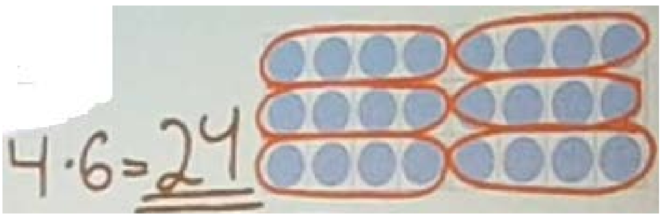


Figure 4. Six groups of four dots in the quick image.

When invited to the board, this student said “four times six” and wrote 4 × 6 = 24. The instructor did not focus on the Norwegian convention (6 × 4), but built on the student’s idea by asking the students how to split the 6 in 4 × 6 into “another multiplication task.” One student answered “two times three” and the instructor wrote 4 × (2 × 3) on the board and said that she just realised that 4 × (2 × 3) could not be represented as the six fours in the image (Figure 4). From what follows, the instructor obviously was thinking of 2 × 3 as six dots, as on dice and was confused. She drew six in a 'clean' quick image as seen in Figure 5.

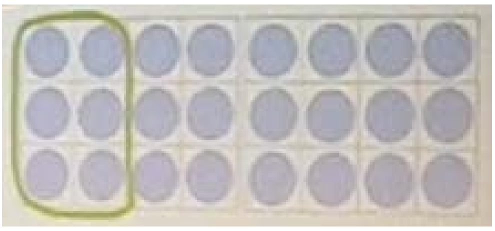


Figure 5. Six dots as represented on dice.

The supervisor asked for a timeout saying: "I think I see the six in the upper image (Figure 4)", and asked the students if they "see six" in Figure 4. They pointed to the six fours, but the instructor did not elaborate on the fact that six in this quick image could be represented as six dots (Figure 5) as well as six groups of four dots (Figure 4). Instead, in the last four minutes of the enactment the instructor circled (4 × 3) × 2, 4 × (2 × 3) and (4 × 2) × 3 on the board saying that they "all have three factors." Factor, factorising and product were discussed and explained, before the instructor pointed to the circled multiplication tasks with three factors and asked the students to discuss what they saw. One of the students saw "that all have three numbers and that all have parentheses and that all [factors] have 'times [the multiplication sign] in between [them]'". Another student added that "all have three numbers" and a third student added that "the multiplication tasks have the same numbers, but they [the factors] have changed positions." This ended the enactment phase.

To summarise, the analysis of the enactment illustrated some components of ambitious teaching practices for teaching multiplication made available for teachers' learning when enacting a quick images activity. In particular, *mathematical work on representations* and *mathematical language* in relation to strategies (*mathematical ideas* involved) initiated by the students were visible. In the enactment of the lesson, the associative property of multiplication (*working towards a mathematical learning goal*) was somewhat invisible. Writing 6 as 2 × 3 and 8 as 4 × 2 in order to obtain three factors was, however, discussed. Towards the end of the enactment phase the associative property was not mentioned, but it was clear that this property was implicitly the focus of attention when the instructor circled (4 × 3) × 2, 4 × (2 × 3) and (4 × 2) × 3.

Discussion after enactment

After the enactment, all the participants looked at the smartboard and focused on the different student strategies represented in the quick image (Figures 2, 3, 4, 5). They were particularly interested in the mathematical idea presented by the student from the enactment who "saw a box with four across and three upwards, then there were two boxes and I took [multiplied] eight times three", writing 8 × 3 = 24 on the board (see Figure 2a). This indicates that the participants preferred that the students chose to use an equal groups model and not an area model (Barmby et al., 2009; Greer, 1992).

From this student idea, they agreed that the students had difficulties in splitting the eight, making the *mathematical representation* in the quick image to arrive at (4 × 2) × 3. The supervisor elaborated on this idea by challenging the instructor with the following question: "How could you have used his [the student's] idea to get what you were aiming for [three factors]? Because he [the student] has actually said it, two groups with four", pointing to the student's illustration in the quick image (Figure 2a). It appeared as if the supervisor was trying to build on this idea, inviting the instructor, but also the other ISTs, to focus on their *mathematical learning goal* for the lesson and how they worked in the enactment towards this goal for the students' learning. But, this initiative was not followed up. Instead, they started to discuss the idea presented by another student in which her strategy was to see four six times in the quick image, writing 4 × 6 = 24 on the board (Figure 4). They discussed how to "split the six", reiterating this situation from the enactment, trying to understand the student's idea. The supervisor made it clear that this student really had a good strategy: "I think she was very clear on that, that she saw four in each group, that she saw six groups and that she saw three times two groups. What she explained was beautiful". IST2 highlighted that it was necessary to circle six dots because "in order to split the six [into 2 × 3] the one you draw (i.e. Figure 5) is the [only] one [possible]". This IST might not see that 2 × 3 could represent two times three groups of four dots (Figure 4). The supervisor concluded that the number six could be represented in the quick image in different ways, depending on whether six is seen as the number of groups or as dots within a group. After this clarification by the supervisor, more ISTs said that they saw the two different ways six could be represented and that 2 × 3 could be six dots or 24 dots (if the 3 represented three groups of four dots). From this part of the discussion the participants had the opportunities for learning how to represent student responses in the quick image (*mathematical work on representations*).

After having discussed these two ideas from the enactment, the participants put much effort into discussing mathematical terms and concepts (05:30–19:56), showing that the component *mathematical language* was prominent. The instructor posed many questions related to the commutative and associative property which were illustrated in the introductory vignette. These questions triggered further discussion on these terms. The supervisor explained and illustrated the associative property by pointing at 4 × (2 × 3) and (4 × 2) × 3 on the board, while IST7 explained the communicative property by going to the board and writing (2 × 3) × 4 and 4 × (2 × 3). Pointing to what the supervisor had just written, he pointed out that these "will be the same", not seeing that this example illustrated the commutative property of multiplication rather than the associative property. The lack of an explicit focus on associativity is understandable since both the instructor and other ISTs had not fully understood the associative property of multiplication (e.g., introduction). The two properties were also not discussed in-depth in this discussion phase of the cycle of investigation and enactment. The analysis of this phase indicated that the ISTs have opportunities for learning in the following areas: *setting and working towards a mathematical learning goal for the students*, *mathematical language* and *mathematical work on representations*. Failing to separate the commutative property from associativity can also be seen as an opportunity for learning (Heyd-Metzuyanim et al., 2015) which was not grasped by the participants.

Concluding Discussion

From our analysis of a cycle of enactment and investigation, components of ambitious teaching practices for teaching multiplication through a quick images activity (Figure 1) made available for ISTs' learning were shown (including opportunities for failing, see Heyd-Metzuyanim et al., 2015). One component was identified when the participants were with addressing the mathematical learning goal for the students. We identified a lack of focus on mathematical learning goals even if the ISTs were challenged four times by the supervisor to come up with such a goal in the planning session. The ISTs struggled to set and work towards a clear ambitious mathematical learning goal for the students, and they seemed to struggle in understanding the mathematical ideas/concepts involved (e.g., the commutative and associative properties of multiplication and the relation between the two properties). The analysis suggests that these struggles created opportunities for the participants' learning in this cycle of enactment and investigation, even though the opportunities were not always grasped by the participants. The mathematical language that is used when teaching quick image as an instructional activity seemed to be challenging for the ISTs, and the supervisors and ISTs appeared to struggle with the mathematical work of representing students' mathematical ideas on the board. However, our analysis also indicated that these struggles created opportunities for the participants to learn components of ambitious teaching practices in the cycle (Ghousseini, 2017; Heyd-Metzuyanim et al., 2015).

Throughout the cycle, a second component of ambitious teaching practices the participants had opportunities to learn was to deal with the *mathematical ideas (strategies and concepts) involved*. This category was related to the ISTs' discussion on *predicting student ideas,* both from testing the activity in their own classroom (planning), but also with respect to discussions (planning and rehearsal) in which they looked at patterns in the quick image to anticipate possible student ideas in the enactment phase. *Prediction* was thus a third component of ambitious teaching practices the participants had opportunities to learn in the cycle of enactment and investigation. Within the Lesson Study literature, research has emphasised the importance of involving predictions of student thinking, particularly when writing detailed lesson plans for a research lesson (Fujii, 2016). This is also an important component of ambitious teaching practices for teaching multiplicative properties that can make it possible for ISTs to learn from the quick images activity. The predictions were related to the anticipated student solutions on the quick image.

A fourth component of ambitious teaching practices the participants had opportunities to learn was illustrated by the *mathematical work on representations*. The analysis revealed how mathematical work on representations was linked to mathematical ideas focused on showing a convenient representation in the quick image. Previous research has pointed out the importance of dealing with mathematical objects through signs and semiotic representations (Lemke, 2003). If students are to understand mathematical concepts (e.g., the notion of a mathematical function), they must learn to "synthesize representations in three different semiotics: verbal language, graphical diagrams, and algebraic expressions?" (Lemke, 2003, p. 226). Our analyses have illustrated how the quick image initiated a discussion among the participants in which these three semiotic representations were based on predicted ideas from students in the ISTs' own classrooms and from anticipated student patterns that could emerge in the enactment session. Building on these ideas from Lemke (2003), we have particularly shown how mathematical ideas represented in a quick image were expressed in a *mathematical language* (fifth opportunity) throughout the cycle (planning, rehearsal, enactment, discussion). The participants dealt with mathematical concepts and tried to express these concepts (the commutative property and the associative property of multiplication) in a proper mathematical language.

In conclusion, the present study investigated components of ambitious teaching practices (e.g., Lampert et al., 2010) made available for ISTs so they can learn to operate in a cycle of enactment and investigation (see methods section) by using a (quite simple) quick image as the instructional activity. The five identified components were:

- setting and working towards a mathematical learning goal for the students,
- understanding the mathematical ideas (strategies and concepts) involved,
- predicting student strategies,
- understanding the mathematical language involved, and
- learning how to represent students' ideas (mathematical work on representations).

Our analysis has shown the complexities of learning to teach elementary mathematics, such as the associative property of multiplication (Schliemann et al., 1998), as well as the opportunities these complexities create for teacher learning. Bearing the findings from this study in mind, we conclude that it is important to provide ISTs with opportunities to work on and discuss the complexities of elementary mathematics as part of their PD. Moreover, it is important that the ISTs are given opportunities to learn to enact components of ambitious teaching practices focusing on these complexities of teaching elementary mathematics, in this study illustrated by using a quick image as an example. We are aware that the sample is small and that small in-depth studies have clear limitations. But, it is our contention that studies like this one are rich when it comes to what is revealed about mathematical thinking related to multiplicative properties. The findings highlight the potential complexities of learning to distinguish and understand commutative and associative properties.

Implications

One implication for future research is the need to study a group of ISTs in several cycles of enactment and investigation (Lampert et al., 2013) and to study several groups of ISTs to delve deeper into the complexities of teaching elementary mathematics to better understand components of ambitious teaching practices made accessible for ISTs' learning in such cycles. Based on the finding that the participants (e.g., in the planning phase) struggled to separate the commutative property of multiplication (i.e., $(a \times b) \times c = c \times (a \times b)$) from the associative property (i.e., $(a \times b) \times c = a \times (b \times c)$), a second implication is the need to study ISTs' mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008; Ghousseini, 2017) parallel to their participation in cycles of enactment and to explore the mathematical complexities of teaching elementary mathematics. A third implication is the need to study similar cycles in new contexts, such as initial teacher education (Kazemi & Wæge, 2015). In their analysis of teacher timeouts in rehearsals, as well as in enactment highlighted as a context for learning, Gibbons et al. (2017) characterised this tool and analysed its potential for supporting the participants' collective learning and development. Teacher timeouts were also used in the MAM project, in rehearsals as well as in enactment.

Future research should seek to understand if and how teacher timeouts support the ISTs in developing their ambitious teaching skills. Since the supervisor is an important part of the discussions throughout the analysed cycle, the findings from this study highlight the important role the supervisor plays in stimulating and facilitating the ISTs and creating opportunities for learning components of ambitious teaching practices. More specifically, the supervisor posed open questions (e.g., related to setting mathematical learning goals for the students) and elaborated on student initiatives from the enactment, inviting the ISTs to take part in the mathematical discussion related to the associative property. The analysis also illustrated the open questions from IST6 who was wondering about and reflecting on the difference between the commutative and associative properties, a struggle this IST seemed to share with several of the other participants. Bearing this in mind, a fourth focus for future research is to examine what can be learned from exploring the discussion from this group of ISTs (e.g., from the fourth cycle), as well as other groups and cycles, investigating how possible dialogic learning communities of interthinking (Littleton & Mercer, 2013) may be created in the sessions. The ISTs and their supervisor appear to act together (interact) and think together (interthink), discussing mathematical properties of multiplication.

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