Developing Pre-service Teachers’ Knowledge for Teaching in the Early Years: Selecting, Sequencing and Assessment

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Developing pre-service teachers’ (PSTs’) knowledge of pedagogical practices can be particularly challenging within university classrooms. Teacher educators are well placed to provide PSTs with theoretical perspectives on pedagogical practices, including those involving assessment; what is particularly challenging, however, is linking theory with practice and developing PSTs’ breadth and depth of knowledge in mathematical concepts. This paper describes the experiences undertaken by two cohorts of PSTs during their tutorials designed to assist them to notice and discuss Year 2 students’ responses to an array task. Open-coding was used to analyse PSTs’ selection and sequencing of five different work samples and their subsequent design of an assessment rubric. Findings indicated that while the authentic work samples assisted the PSTs to make connections with the students’ mathematical understandings, the lesson also provided an insight into the PSTs’ own foundation knowledge of mathematical understandings relating to children’s development of multiplicative thinking.

**Keywords.** preservice teachers **.** formative assessment **.** teacher knowledge **.** multiplicative thinking **.** early years

# Introduction

Reforms in mathematics education require different approaches to teacher education that involve a shift in the kind of learning experiences currently provided for prospective teachers (XXXX xxxxx). PSTs value university experiences that have particular classroom relevance (e.g., Beswick, 2006) with practice-based strategies being effective in bridging the perceived gap between theory and practice (XXXX XXXX). Bridging this gap can be challenging, and while theories may help us to analyse or explain, they do not necessarily provide guidance for practice (Jaworski, 2006). According to Klein (2012), it is time for PSTs to move beyond knowing and understanding mathematical ideas; they need to learn to teach mathematics they most likely have not experienced, and learn to teach in innovative ways. Representations of practice (Grossman, Hammerness, & McDonald, 2009), such as children’s work samples, have the potential to support PSTs to learn not just about teaching, but how to use knowledge of teaching in action (Grossman, et al., 2009). Other evidence suggests that primary PSTs’ capacity to identify errors in children work samples is clearly linked to levels of mathematical content knowledge (MCK) (XXXXX xxxx).

According to the Teacher Education Ministerial Advisory Group (TEMAG), universities need to improve the way in which they support PSTs’ pedagogical practices and mathematics knowledge (TEMAG, 2015). Research has indicated that overall teacher knowledge is linked with teaching efficacy (Beswick & Goos, 2012), which may be particularly relevant to early childhood educators who often have negative attitudes towards mathematics and feel underprepared to teach mathematical concepts (Cohrssen & Tayler, 2016). Knowledge for teaching includes knowing how and what to teach but also relies on the teacher’s ability to design, select and analyse worthwhile mathematical tasks (Sanchez & Garacia, 2011). Once the tasks are chosen the teacher is then responsible for facilitating children’s learning throughout the lesson and may employ particular instructional strategies. Smith and Stein’s (2011) identification of five practices that promote mathematical discussion provide a framework that can be used to facilitate children’s mathematical learning. Discussed later in this paper, the practices of selecting and sequencing work samples are particularly relevant to this study. The lesson and approach documented in this paper was designed to assist PSTs to examine children’s multiplicative thinking as evidenced by work samples. In particular, the teacher educators were interested in the PSTs’ discussion, selection and sequencing of children’s work samples generated from a multiplication task, and subsequent design of a related assessment rubric. Data collected from the PSTs were examined to address the following questions:

What do PSTs identify as important when selecting and sequencing work samples to share during an early years’ mathematics lesson?

What PSTs’ knowledge of multiplicative thinking is evident when analysing children’s work samples and designing qualitative descriptions within an assessment rubric?

In what ways do student work samples and rubric construction support PSTs knowledge of student thinking?

The focus on multiplicative thinking was considered appropriate as it is a crucial stage in children’s mathematical understanding, and underpins mathematical concepts such as proportional reasoning, ratio and algebra (Dole, 2008). The shift from additive thinking to multiplicative thinking occurs from Years 2-4, therefore it is vital that early years’ educators understand different multiplication structures and how to teach these concepts to children.

# Review of Literature

## Teacher Knowledge

Various frameworks exist and have been used to understand the different types of knowledge required for teaching (e.g., Ball, Thames, & Phelps, 2008; Chick, Baker, Pham & Cheng, 2006; Rowland, Turner, Thwaites, & Huckstep, 2009). Shulman’s (1987) identification of the different knowledge types required for teaching has resulted in a body of literature, examining teacher and PSTs’ knowledge, including MCK and Pedagogical Content Knowledge (PCK). A review of the literature confirmed that PSTs’ MCK was of concern and a topic of interest nationally and internationally (e.g., Callingham, Beswick, Chick, Clark, et al., 2011; Tatto, Schwille, Senk, Ingvarson, et al., 2012).

Teachers of mathematics, including early years’ teachers, rely on their PCK and MCK in many contexts. PCK is central to understanding children’s misconceptions; knowing how topics are organised and taught; and having the ability to adjust lessons to cater for all learners (Shulman, 1986). Building on the seminal works of Shulman, content knowledge or MCK has been described within different frameworks. Ball et al.’s (2008) domains of mathematical knowledge for teaching used three categories to describe the subject matter teachers require. Teachers rely on common content knowledge when planning and teaching (Ball & Bass, 2003; Thames & Ball, 2010) and can shape the way they use their PCK (Shulman, 1986). They also rely on *specialised content knowledge* that is unique to teaching (Ball et al., 2008). When demonstrating specialised content knowledge, they may listen to and interpret student responses; analyse student work; and decide what to clarify when teaching (Ball & Bass, 2009). The third category *horizon content knowledge* (Ball et al., 2008) or *knowledge at the mathematical horizon* (Hill, Ball, & Schilling, 2008), is a type of peripheral vision that informs teachers’ practice when understanding the complexities of mathematical topics. Horizon Knowledge is similar to Ma’s (1999) notion of Profound Understanding of Fundamental Mathematics (PUFM) manifested when teachers make connections with topics they teach demonstrating *breadth* and *depth* of knowledge.

The Knowledge Quartet framework considered the different kinds of knowledge that teachers need for teaching and was developed from observations of 24 mathematics lessons (Rowland, et al., 2009). The Knowledge Quartet contains four dimensions: *foundation knowledge*, *transformation*, *connection*, and *contingency* and 18 codes useful for interpreting classroom practice (Table 1). While the Knowledge Quartet was designed to be used as a framework for identifying ways MCK was enacted in practice, it was used in our study to interpret PSTs’ responses to two different tutorial activities.

Table 1

The dimensions and associated codes of the Knowledge Quartet (Rowland et al., 2009, p. 29)

|  |  |
| --- | --- |
| Dimensions | Codes |
| Foundation knowledge | Adheres to textbookAwareness of purposeConcentration on proceduresIdentifying errorsOvert subject knowledgeTheoretical underpinningUse of terminology |
| Transformation | Choice of examplesChoice of representationDemonstration |
| Connection | Anticipation of complexityDecisions about sequencingMaking connections between proceduresMaking connections between conceptsRecognition of conceptual appropriateness |
| Contingency | Deviation from agendaResponding to children’s ideasUse of opportunities |

A recent review by Turner (2012) matched categories of the Knowledge Quartet with other frameworks including Shulman, Ball and colleagues. The dimension of *foundation knowledge*, relates to the knowledge the teacher has learnt at school or as part of their teacher education (Turner & Rowland, 2011). Turner (2012) described features of *foundation knowledge* as concerning subject knowledge and PCK (Shulman, 1987) and also *common content knowledge* and *specialised content knowledge* (Ball, et al., 2008). The dimension, *connection*, refers to the choices and decisions a teacher makes and the ‘coherence’ displayed when planning or teaching (Turner & Rowland, 2011). When teachers demonstrate *depth* of knowledge they make connections with a topic of greater conceptual power (Ma, 1999). The dimensions of *foundation* and *connection* were particularly relevant to the study discussed in this paper, whereas *transformation* and *contingency* relate more readily to observing teaching (Turner & Rowland, 2011).

While *foundation knowledge* differs from the other dimensions in the sense that it is about knowledge possessed, rather than the application of knowledge, *connection* requires consideration of a number of aspects including making links with prior learning and different areas of mathematics, and anticipating complexity. The reform agenda for mathematics education with its emphasis on developing conceptual understanding, highlights the importance of connected knowledge for teaching (Ball, 1990), and Askew, Brown, Rhodes, Johnson, and Wiliam’s (1997) seminal study found that the most effective teachers of numeracy were those who emphasised connections, rather than teaching and practising standard methods in isolation. Making connections between concepts links to Ma’s description of *breadth* and might be evident in a multiplication context, for example, when teachers (or children) know more than one way to model an array for 24 such as 2 rows of 12 or 6 rows of four. Making connections also requires a *depth* of knowledge and *horizon knowledge* and may involve, for example, the teacher having a capacity to work flexibly with concepts, strategies and representations of multiplication at and beyond the level they teach.

## Multiplicative Thinking

A recurring theme in the literature is that multiplicative thinking is a crucial stage in children’s mathematical understanding, the basis of proportional reasoning, and a necessary pre-requisite for understanding algebra, ratio and rate, and scale, and interpreting statistical and probability situations (e.g., Dole, 2008; Siemon, Bleckly & Neal, 2012). Siemon et al. argue that teachers need to be aware of the ideas and understand the strategies underpinning the transition from additive to multiplicative thinking. XXXX xxxxxxx share this view and suggest that early years teachers need to have an understanding of the different multiplicative semantic structures and be prepared to engage children in challenging tasks related to these different structures.

The transition to multiplicative thinking is conceptually demanding as a child needs to coordinate a number of equal sized groups and recognise the overall patterns of composites, such as 'three sixes’ (Clark and Kamii, 1996; Steffe, 1994). A composite is defined as “a collection or group of individual items that must be viewed as one thing” (Sullivan, Clarke, Cheeseman, & Mulligan 2001, p. 234); that is, conceptualising three as one unit of three, rather than three single ones. It is important that teachers understand the complexity associated with this shift and the level of abstraction required, compared to additive thinking. Steffe (1994) described the demands of multiplicative thinking as:

For a situation to be established as multiplicative, it is necessary at least to co-ordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit (1994, p. 19).

A key issue for teachers is whether it is possible to promote the move from additive to multiplicative thinking with children and how this might be done. Sullivan, et al., (2001) suggested that abstracting, characterised by students moving beyond the need to create physical models, to forming mental images to find solutions, was a key stage in the learning of multiplicative concepts. Singh (2000) found that when students move from additive thinking to multiplicative thinking with whole numbers, two important changes occured, the first being a shift from “operating with singleton units to coordinating composite units” (p. 273), and the second a change in the meaning given to a number.

Historically the teaching of multiplication has focused on the equal groups model in which children accumulate groups of equal size to represent a situation. This however, is an additive model that encourages children to think of four threes as 3 plus 3, plus 3 plus 3. A consistent theme in the literature is that children require experiences with different multiplicative situations such as rectangular array/area, multiplicative comparisons, allocation/rate, when supporting development of multiplicative thinking (e.g., Anghileri, 1989; Greer, 1992). The rectangular array structure in particular, is powerful and provides a visual representation of the mapping of two spaces into a third (Vergnaud, 1988), assists students to develop a sense of the relationship between the numbers, and flexible partitioning of numbers (Young-Loveridge, 2006). The physical arrangement of objects makes the property of commutativity, intuitively recognisable (Greer, 1992) and assists generalising the factor, factor, product relationship, that is important for supporting fraction representation (Hurst & Hurrell, 2014).

Being able to form visual images of the composite unit structure is essential to making the transition to multiplicative thinking (Sullivan et al., 2001). Interpreting the structure of arrays requires spatial visualisation processes such as visual imagery. For instance, students need to combine both spatial (rows of squares) and numeric composites (number of squares in a row), when constructing an array (Battista, 1999). Mulligan, Prescott, and Mitchelmore (2006) found that spatial structuring played a key role in visualising and organising multiplicative structures such as unitising and partitioning. Battista, Clements, Arnoff, Battista, and Borrow (1998) argued that in order for students to make sense of rectangular arrays and use multiplication to enumerate the objects they needed to see a row-by-column structure.

Having an understanding of the concepts underpinning multiplicative situations assists teachers to interpret children’s intuitive strategies. Research indicates that pre-school children can solve a variety of multiplication problems by combining direct modelling with counting and grouping skills, and with strategies based on addition (e.g., Anghileri, 1989; Clark & Kamii, 1996; Mulligan & Mitchelmore, 1997). The progression to multiplicative thinking begins with modelling (using physical objects, fingers, or drawings) progressing to some form of calculation (e.g., unitary counting, to skip counting) often still accompanied by materials, to additive strategies based on repeated addition, and finally to multiplicative strategies, such as known and derived multiplicative facts (Anghileri, 1989; Mulligan & Mitchelmore, 1997). This knowledge is necessary when teachers observe children when engaged in a task and making informed decisions within the moment in a lesson.

The XXXXX task used in this study was part of a series of tasks that were designed to develop students’ multiplicative thinking in the primary years of schooling (Sullivan, 2016). As previously mentioned, children in the early years are often introduced to multiplicative situations with the use of arrays. This focus is also reflected in the Australian Curriculum: Mathematics (AC: M) where multiplication is explicitly referred to in the Year 2 descriptor: Recognise and represent multiplication as repeated addition, groups and arrays (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2016).

## Pedagogical Practices

Clarke, Grevholm and Mimann (2009) agreed that teacher education courses should provide productive practices that integrate learning of knowledge for teaching. This section provides an overview of productive pedagogical practices that are particularly relevant to our study.

Smith and Stein’s (2011) framework provides five practices that could be used by teachers to stimulate mathematical discussions including:

*Anticipating* likely student responses to challenging mathematical tasks;

*Monitoring* students’ actual responses to the tasks (while students work on the tasks in pairs or small groups);

*Selecting* particular students to present their mathematical work during the whole-class discussion;

*Sequencing* the student responses that will be displayed in a specific order; and

*Connecting* different students’ responses and connecting the responses to key mathematical ideas (Smith & Stein, 2011, p. 8).

The two practices of *selecting* and *sequencing* were explored in our study. While a common classroom strategy involves students’ sharing their work with the class, the practice of *selecting* particular students and their solutions to share is made strategically rather than randomly or voluntary. Once selected, the teacher makes purposeful choices about the *sequence* in which to share these solutions. By carefully considering the *sequence*, teachers can maximise opportunities for children to learn. Often the choice of *sequencing* begins with the least sophisticated representation or strategy and ends with a more sophisticated solution (Smith & Stein, 2011). However, this can vary in practice, and *sequencing* decisions may be made on the basis of sharing the most common strategy, highlighting particular misconceptions, or comparing contrasting strategies. Smith and Stein (2011) acknowledge that more research needs to be undertaken to compare the value of different sequencing methods, and this paper adds further to our understanding of this.

A comparable study undertaken by Meikle (2016) investigated American, middle year PSTs’ *selection* and *sequencing* of student solution strategies. Meikle’s PSTs were given six anticipated solution strategies to a fraction task and asked to provide a rationale for the way they selected and sequenced the tasks. Meikle found the PSTs selected solution strategies for both pedagogical and mathematical reasons and generated the following three categories for interpreting their responses:

Pedagogical Moves: may focus on solution with a misconception, incomplete or contained abstract representations;

Mathematical Procedures: made reference to mathematical procedures that related to procedural knowledge rather than conceptual understanding; or

Underlying Concepts: making connections back to the learning goal of the lesson.

Meikle (2016) noted that some students provided rationales that focused on pedagogical moves and failed to include deeper consideration of the mathematical learning goals. However, her PSTs were more likely to focus on mathematical procedures when selecting children work samples than other mathematical aspects

### Challenging tasks

In order to engage in productive discussions, Smith and Stein (2011) advocated that teachers need to have clear learning goals for the lesson and must select a task or tasks that have the potential to help children achieve those goals. Current theories for teaching include the use of highly cognitive demanding tasks or challenging tasks, which are designed to promote rich student-centred learning (Sullivan, Cheeseman, Michels, Mornane, Clarke, et al., 2011). Sullivan, Aulert, Lehmann Hislop, Shepherd et al., (2013) listed some key characteristics of challenging tasks as requiring “students to connect different aspects of mathematics together, to devise solution strategies for themselves and to explore more than one pathway to solutions” (p. 618).

The task from which the work samples were produced originated from the XXXX project. This project provided participating teachers with support in implementing appropriate challenging tasks and experiences with their children. The XXX tasks were designed so that children would engage with important mathematical concepts, attempt the task by themselves, record and explain their thinking (Xxxxxxxxxxxx xxxxx).

As part of the project, teachers were encouraged to adopt a three-stage process for teaching with challenging tasks: launch, explore, and summarise (Sullivan, Boreck, Walker & Rennie, 2015). During the explorephase the teacher considers possible ways of sequencing the children’s responses to highlight the key mathematical ideas, solutions and strategies. These are then usually shared in the summary phase of the lesson when the teacher draws together the learning that has occurred using the strategies and solutions from the children (Sullivan, et al., 2015). The first author used this approach with a Year 2 class to generate the authentic work samples used in the study.

### Assessment

The use of work samples of open ended tasks are often used by practicing teachers to assess children’s thinking and to provide evidence of understanding for both formative and summative assessment purposes (XXXXX xxxx; Yeo, 2011). Selecting and sequencing these work samples requires careful consideration, particularly when constructing task specific assessment rubrics. There is growing evidence in the research literature (e.g., Jonsson, & Svingby, 2007; Yeo, 2011) that the use of rubrics adds to the quality of formative and summative assessments. A rubric can be thought of as a set of criteria for assigning scores in making a holistic judgement on student learning based on the quality of student written work on a specific task (XXXXXX xxxx). The focus within a holistic rubric is what students know and can do, and the mathematical knowledge that is apparent from the child’s response. It enables the teacher assessing a student’s work to determine the evidence of students’ understanding and the communication of that understanding (Yeo, 2011). Doing so focuses on the key mathematical ideas underpinning the tasks rather than recall of facts and procedures. Marzano (2002) maintained that task specific rubrics were more likely to produce more generalisable and dependable scores than a generic rubric.

In summary, teacher educators were tasked with the challenge of developing PSTs’ mathematical knowledge for teaching and preparing them for entry into the profession. Studies have shown that university experiences, which have classroom relevance, can be particularly effective in making links between theory and practice (e.g., XXXX XXX). Hence our focus on providing our PSTs with opportunities to engage in authentic assessment practices. The review of the literature has highlighted how mathematical knowledge for teaching can be considered within the framework of the Knowledge Quartet (Rowland, et al., 2009), along with the *depth* and *breadth* of knowledge required to understand the concepts involved in multiplicative thinking. The next section details the methodology that was adopted for the study before considering the results that were generated.

# Method

The aim of the qualitative research reported in this paper was to observe PSTs’ approaches to analysing five mathematical work samples produced by Year 2 children. In order to gain insights into what the PSTs noticed about the work samples, they were asked to sequence the order they would share the samples with a hypothetical class and justify the reasons for their choice. To provide additional insight into their understanding of children’s multiplicative thinking as evidenced by the work samples, the PSTs were also asked to design a marking rubric that would be suitable for assessing the children’s responses. A case study method was adopted as case studies are appropriate when “you [want] to understand a real-life phenomenon in depth, but such understanding encompass[es] important contextual conditions – because they [are] highly important to your phenomenon of study” (Yin, 2009, p. 18). In this instance, the phenomenon to be examined was the PSTs’ approaches to the task, with the context being a hypothetical Year 2 classroom.

## Participants

The participants were two cohorts (N=49) of third-year undergraduate primary PSTs enrolled in their second and final core subject for primary mathematics teaching. The first core primary mathematics unit focused on early years’ mathematics pedagogy including an introduction to early multiplication concepts such as arrays. The third-year unit focused on primary teaching in the middle years. PSTs had limited experience of assessment practices related to analysing children’s work samples and constructing rubrics. Both units were held over a semester and included two hours of face-to-face tutorials and one hour of reading and online activities each week for 10 weeks. Some of the PSTs (n=9) were also completing a major in primary mathematics education and during their course completed an additional six units related to primary mathematics teacher education, in other words two units during first, second and third year per semester.

The first cohort (n= 29) responded to the selecting and sequencing activity and the second cohort (n=20) of PSTs responded to the selecting and sequencing activity as well as designing a rubric activity. (The first cohort completed the assessment rubric but these data were not collected.) An additional two PSTs, in cohort two completed the activities but chose not to submit their data for the study.

## Context and Instruments

Prior to data collection, the first author taught a multiplicative lesson at the end of 2015 to a mixed ability class of 27 Year 2 children. The first author had not visited the school before and therefore had little knowledge of the children’s prior learning. The classroom teacher mentioned that the class had completed work on early multiplication skills and had been exposed to arrays. Some of the children were able to recall multiplication facts such as, “three times five is fifteen or two times ten is twenty.”

Before commencing the lesson the children were asked to sit in a circle and each take 12 counters, make an array and then share their different responses. For example, three rows of four or two rows of six. Next the children were asked to consider the following array task (Sullivan, 2016), “I had a full box of chocolates, but someone ate some of the chocolates. The box now looks like this (see, Figure 1), how can I work out the number of chocolates I started with?”

In keeping with the ethos of the XXX project (XXXXXXX) the children were given a sheet with the problem, asked to attempt the task individually, and no further instructions were given.



*Figure 1.* Array Task (Sullivan, 2016)

After the lesson the work samples were collected. Five of the children’s work samples were chosen as instruments for this study (Figures 2, 3, 4, 5 and 6). The samples were selected to show a range of different examples of multiplicative thinking. The work samples are presented next, including a summary of the student response and justification for selection as instruments.



*Figure 2.* Sample A

In Figure 2 the child wrote, “I worked it out by cating [counting] in fives to now [know] that it is 45. I used fives to cont [count] it because it is easyer [easier].”

The child recorded their thinking without reference to a drawing or an array, suggesting he was visualising how many chocolates might fill the box. This work sample was chosen to illustrate the transition from concrete to visual thinking. The use of skip counting suggests that he understands the equal group structure.



*Figure 3:* Sample B.

Figure 3, the child drew the chocolate box and wrote, “by counting and imagineing [imagining] there were still chocolate in the box.”

This work sample suggests the child was able to visualise their answer by drawing an array that matches her number sentence and explanation that shows her thinking. The work sample was chosen because it showed the connection between a diagram, equation and explanation of her thinking. Sample B also showed an understanding of the row and column structure of an array.



*Figure 4.* Sample C.

In Figure 4 the child included a drawing of the chocolates and four number sentences for 24 “$4×6=24, 6×4=24, 24×1=24$ and $1×24=24$,” as well as a drawing of “4 grops [groups of five].

Sample C showed some of the child’s understanding of commutativity and symbolic notation of multiplication. He correctly recorded number facts for 24. This sample was selected because it showed both the array structure and equal group structure representations for multiplication.



*Figure 5.* Sample D.

In Figure 5 the child attempted to draw 8 rows of 11 chocolates and recorded 8 $×$ 11 = 76 and 11 $×$ 8 = 76

Sample D shows the child was able to record an attempt to draw 8 rows of 11 chocolates but she recorded an incorrect multiplication number sentence that did not match her array. The sample also demonstrates knowledge of commutativity because she recorded $8×11=76$ and $11×8=76$ although the calculation was incorrect.

The sample was chosen because the child made a reasonable attempt at drawing an array but did not attend to the importance of having equal columns and equal rows. In addition, the response did not reflect a realistic solution. The sample provides potential to gain insight into how PSTs might react to an incorrect response.



*Figure 6*. Sample E.

Figure 6 shows a row of five and four chocolates and two number sentences 3+6=9 and 6+3=9. The child attempted to fill the box by adding six chocolates to the three chocolates shown in the problem and recorded a number sentence to reflect this, 3 + 6 = 9 and 6 + 3 = 9. The second number sentence 6 + 3 = 9 suggests some understanding of the commutative property for addition. This response indicates limited understanding of an array or the meaning of multiplication. The final sample was selected as it shows naïve knowledge of the array structure and the concept of multiplication. The authors hypothesised that the PSTs might not select this sample.

## Procedure

The following outlines the procedure for the array and rubric activities. For both activities, qualitative data were collected in the form of PSTs’ written responses. Before the selecting and sequencing activity the PSTs were asked how the children might answer the array task (Figure 1). For the array activity, the PSTs were asked to individually select three of the five work samples (Figures 2, 3, 4, 5, and 6). They were then asked to imagine being a Year 2 teacher and to select and sequence three of the children’s work samples they would choose to share as part of the summary phase of the hypothetical lesson, without discussing with their peers. They recorded their responses on a proforma with a copy of the children’s work samples. Following this they were asked to explain and justify their choices as part of a class discussion. Their written responses were collected for analysis. The array activity took approximately 40 minutes to complete.

Following the selecting and sequencing activity the second cohort of PSTs worked in pairs to design an assessment rubric suitable for assessing children’s responses to the array task (Figure 1), using a template designed by the second author. The PSTs recorded five responses summarising their descriptions for the categories of content and processes. Content descriptions reflect the level of children’s mathematical understanding; process describes the strategies and the reasoning evident within the children’s work samples considering the proficiencies of the AC: M (ACARA, 2016). In order to assist them they were provided with a generic scoring rubric (see, Table 2). Following the generation of the assessment rubric, the PSTs then matched the five children’s work samples to a score using their rubric, and asked to respond to the following reflective questions:

What aspects of the task did you find challenging (if any)?

What assisted you when completing this task?

The authors (one and two) collected the PSTs’ responses at the end of the class. Both authors recorded field notes which were discussed after the lesson. The PSTs took approximately 60 minutes to complete their summary descriptions.

Table 2

Generic scoring rubric

|  |  |
| --- | --- |
| Score | Summary Description |
| Content | Processes |
| 5Goes beyond | Fully accomplishes the task, but uses methods and/or makes interpretations significantly beyond those specified for this level.  | Strategies/ mathematical communication/ reasoning significantly beyond those specified for this level.  |
| 4Task accomplished | Task accomplished. Central mathematical ideas clearly demonstrated and understood  | Appropriate plan. Clear communication of strategies and mathematics used  |
| 3Substantial progress | Substantial progress towards completing the task; indicative of understanding of relevant knowledge, concepts and skills, but some key ideas may be missing  | Some evidence of planning; some communication of strategies and mathematics used  |
| 2Some progress | Attempt at the task makes some progress; partial but limited grasp of the central mathematical ideas; reveals gaps in knowledge, conceptual understanding and/or relevant skills.  | Little evidence of effective strategies/ communication/ reasoning  |
| 1Little progress | Little progress or understanding evident.  | Ineffective strategies/ communication/ reasoning  |

## Data Analysis

The first two authors coded all data. An open-coding approach (Flick, 2009) was used initially, to identify common themes. The coding was compared and a discussion held about the reasons for any coding differences and agreement reached. The Knowledge Quartet was then used as a framework for organising the coded data, particularly in the dimensions of *foundation* and *connection*. This two-step process was done to ensure that the coders were open to any elements that may have been masked by exclusively using an existing theoretical framework.

### Data analysis of the sequencing activity

When analysing PSTs’ responses to the sequencing activity the authors identified which work sample the PSTs selected first, second and third. The data were tallied and recorded into a table (see Table 5) to compare their number of choices for children’s responses Samples A to E (Figures 2 to 6).

The PSTs’ written responses from the proforma were analysed to identify common themes. The first step involved allocating codes to each of the responses given for the selection of either first, second or third choice (Table 3).

Table 3

Coding of Pre-service Teachers’ Written Responses to the Selecting Activity

|  |  |
| --- | --- |
| Elaboration of coding | Foundation knowledge  |
| Indicated some awareness of key mathematical ideas underpinning the task (visualisation, equal groups structure, array structure) | Overt subject knowledge  |
| Highlighting the different student thinking/strategies they used to record their solution | Concentration on procedure |
| Focused on student misconceptions/incorrect response | Overt subject knowledge Identifying errors |
| Justification provided a general recount of what children did/limited justification or thought of the mathematical underpinnings | Theoretical underpinning of pedagogy  |
| Made assumptions about child’s thinking | Awareness of the purpose (lack of) |
| Showed little or no connection or attention to miscalculation or incorrect representation | Identifying errors (lack of) |
| Correct response  | Concentration on procedures |
| Use of multiplicative language | Use of terminology |

A total of eight codes were generated, including mathematical ideas; strategies; misconceptions; limited mathematical justification; made assumptions; little or no connection; correct response; and multiplicative language. Some responses had more than one code, as evident by the following response which was coded as mathematical ideas and strategies because of reference to visualisation and strategies.

I would choose response B first. I would choose this because it was a good demonstration of how the students could use different strategies to work this out, where this child drew a picture and counted the chocolates he imagined in the box.

The work sample(s) were further coded as *overt subject knowledge* and *concentration of procedures* which are elements of the foundation dimension of Knowledge Quartet Rowland et al., 2009) (Table 1). Note for the purpose of our paper we used the term elements when discussing the codes of *foundation* and *connection* (Rowland et al., 2009).

### Coding and data analysis of the assessment rubric

Two levels of analysis were used when reporting PSTs’ responses to the assessment rubric activity. The first included descriptive analysis to code the descriptors for both content and process. The second level of analysis drew on the elements of foundation and connections contained in Rowland et al.’s (2009) Knowledge Quartet. As an illustrative example, the following content response was coded as ‘multiplicative language’ and ‘mathematical ideas’.

Visualises and illustrates number of chocolates in array, with correct answer. Able to demonstrate a variety of problem representations.

This was further coded as *use of terminology* and *theoretical under pinning* which are elements of the foundation dimension of the Knowledge Quartet (Rowland et al., 2009). In order to code the rubric to the connections dimension of the Knowledge Quartet we looked at specific examples of the elements rather than look at the rubrics as a whole due to the breadth of the elements. A rubric by its very nature involves decisions about sequencing, making connections between procedures; making connections between concepts; recognition of conceptual appropriateness.

### Coding and data analysis of the rubric activity

After data collecting the PSTs’ responses within the scoring rubric template for the five categories of *content* and *process* (see Table 1) were coded identifying PSTs’ mathematical understanding related to multiplicative thinking. For each of the five scores within the rubric the authors coded the PSTs’ collated responses using a scale of (i) (generic and no connection to mathematics within the task) up to (v) (description specifically related to the mathematics and consideration of the child’s thinking beyond expectation). Table 4 provides an overview of the initial coding used when analysing PSTs’ content and process descriptors in their rubric.

Table 4

Coding of Pre-service Teachers’ Descriptions when Responding to the Rubric Activity

|  |  |
| --- | --- |
| Pre-service teachers content coding | Pre-service teachers process |
|  v. Working towards generalising, formulating an original problem, recognises that a variety of problems for a given number may have the same solution (working towards the property of commutativity). | v. Gives examples of strategies beyond this level, using multiplication facts. |
| iv. Acknowledge successful completion of task (but not beyond) and demonstrates a variety of problem representations and understanding of multiplication beyond the required level. | iv. Making connections to children’s strategies and includes examples of student’s strategies and their reasoning and/or use of mathematical language within their description.  |
| iii. Used some mathematical language, concepts beyond grouping and multiplication. | iii. Beginning to make connections to student’s strategies and some indication that children struggled to explain their reasoning.  |
| ii. Generic description with reference to children’s process: mathematical thinking or advanced reasoning. | ii. Some connections but limited to children’s work samples and strategies. |
| i. Generic description no links to the mathematics within the task. | i. Lacking connections to student strategies |

Following the generation of the codes (Table 4), the authors used the codes to analyse PSTs’ responses for all scores of both content and process descriptions. Reporting these results is beyond the scope of our paper. For this study, the codes were used to gain an overall sense of each pair of PSTs’ rubric description and evidence of their *foundation knowledge* and *connection* using further coding from the Knowledge Quartet (Rowland et al., 2009).

# Results and Discussion

## Selecting and Sequencing Activity

The results are presented in three sections; reporting on PSTs’ responses to the selecting and sequencing of Year 2 children’s responses to an array task; reporting on PSTs’ descriptions within a marking rubric they generated to assess the children’s work samples; reporting on *connections* of the Knowledge Quartet (Rowland et al.*,* 2009) evident within PSTs’ written descriptions of their assessment rubrics.

## Reports of the Selecting and Sequencing Activity

The following presents the results of the selecting and sequencing activity. Table 5 shows how many participants selected each of the Samples (Figures 2 to 6) as their first, second or third choices.

Table 5

Pre-service Teachers’ Number of Choices for Children’s Responses (Samples A-E) (n=49)

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Participants | A | B | C | D | E | No choice |
| First choice | 5 | 11 | 6 | 4 | 13 | 0 |
| Second choice | 5 | 12 | 14 | 4 | 4 | 0 |
| Third choice | 3 | 12 | 16 | 4 | 3 | 1 |

As seen in Table 5 there were a number of different suggestions and combinations of selecting and sequencing. In practice, selecting and sequencing student responses requires considered judgements, and it is likely that in the event that a number of teachers were asked to select and sequence a range of students’ responses, they would differ in their choices and the reasons for making them.

One third of PSTs chose Sample E as a first choice and almost another third chose Sample B. This is interesting because Sample E reflects a misconception or limited understanding of arrays, whereas Sample B shows correct representation of the array structure, along with the related number sentence and an explanation of the student’s thinking (Figure 3). The results show the vast differences between the response the PSTs would select as their first choice. When selecting their third choice, seven PSTs selected the “weakest response,” for example Sample A then, Sample D followed by Sample E. The thinking of the other four PSTs was to select, “strong-weak-strong,” hence selecting Sample C as their third choice.

When justifying why they selected Sample B first, one PST wrote:

I would choose this because it is a good demonstration of how the students could use different strategies to work this out, where this child drew a picture and counted the chocolates he imagined in the box.

In contrast one PST who chose Sample B and then Sample C (second choice) did not select a third choice, and wrote, “The rest are so wrong I would not show them to the class.” A possible explanation for her choice was that she was focusing on correct solutions, representations, and/or children’s understanding of the array structure. This PST’s response highlights for teacher educators the importance of discussion of misconceptions or children’s errors with PSTs when building their *foundation knowledge*.

The selection of PSTs’ second choice was predominately Sample C (n= 14) followed by Sample B (n=12). The PSTs may have chosen Samples B or C because both work samples showed correct arrays and related number sentences, however Sample C did not provide evidence of children’s thinking. Similarly, for their third choice, slightly more PSTs chose Sample C (n=16) compared to Sample B (n=12), possibly for the same reasons. To further interpret the PSTs’ thinking, each first choice sample selection will now be discussed in turn.

*Sample A* (Figure 2), was chosen by five PSTs as their first choice. Examples of their justifications included because it provided evidence of how and why the student came to the solution of 45 for the number of chocolates. One PST suggested that a possible discussion point would be to highlight an inclusion of a drawing of the child’s thinking (an array) to enrich the response.

*Sample B* (Figure 3) was chosen as the first choice by 11 PSTs because it was described as the “strongest response.” One PST justified this by stating that it “showed three components, a diagram, equation and an explanation of thinking”. Other reasons were: “… the child could imagine where the other chocolates could be…and drew all the imagined counters,” and, “This work sample showed their working out as well as how they came to their answer.”

*Sample C (Figure 4)* was selected by six PSTs. Examples of the PSTs’ thinking included: “the child was using, grouping thinking to calculate and show different arrays.” When the PSTs were interpreting Sample C at least three PSTs indicated the child was using, “grouping thinking to calculate and show different arrays,” rather than understanding that the groups-of idea is different to the array model.

*Sample D (Figure 5)* was selected as first choice by four PSTs and was their least common first, second or third choice. A reasoning for one PST selecting Sample D first, was the “weakest response” and:

… the student is displaying array knowledge and I think it would clue the struggling students into looking at the array.

Another PST chose this sample as the second choice and explained:

To remind the students of the ‘gap’ or empty space between chocolates, encouraging everyone to think about how the circles relate to the imaginary chocolates… the need to replicate the way the chocolates would fit into the box.

*Sample E (Figure 6)* was chosen by 13 PSTs as their first choice, primarily because they thought it was the weakest response. One PST wrote that she would, “Ask the class if they think this is correct and why it wouldn’t be?”

Another PST who selected Sample E first and wrote:

By explaining this first you can initially eliminate any misconceptions about the box being that size. Students will then be able to (if not already) recognise that there is empty space and therefore more chocolate at the bottom of the box.

Table 6 shows further analysis of the PSTs written responses with relation to the eight codes that emerged from the data.

Table 6

Codes identified within PSTs’ written justifications for the selecting and sequencing activity

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Codes | Example of PSTs response | Sample A | Sample B | Sample C | Sample D | Sample E |
| Mathematical ideas | I believe the child understands 4 groups of 6 and multiplication | 4 | 15 | 16 | 3 | 3 |
| Strategies | The child has set out their thinking… explained it. | 10 | 23 | 30 | 3 | 1 |
| Misconceptions | They used addition to work out the answer [rather than multiplication] |  |  |  | 5 | 11 |
| Limited mathematical justification | Broke down the dots. | 2 | 5 | 1 | 1 | 3 |
| Assumptions | Simple and small number for children. | 2 | 2 | 2 | 2 | 2 |
| Little or no connection | Can help other students get started |  |  |  | 6 | 2 |
| Correct response | Correct answer – students who are strolling would see correct solution |  | 3 | 6 |  |  |
| Multiplicative language | … has used the terms groups of and arrays… |  | 2 |  |  |  |

The table shows that the majority of PSTs attended to student thinking/strategies for Samples A to C followed by evidence of their awareness of the key mathematical ideas underpinning the task. For samples B and C, not surprisingly they identified Sample B as an opportunity to address student misconceptions. Eight of the PSTs’ justifications for choosing Samples D or E did not identify the child’s error or miscalculation. This possibly suggests limited foundation knowledge, MCK and/or PCK.

The specific mathematical ideas evident in the PSTs’ justifications included the use of *visualisation*, *equal groups*, and *array structure* and consideration of *student thinking and strategies*. We suggest that these PSTs were demonstrating their specialised MCK for teaching relating to multiplicative thinking. More specifically, *foundation knowledge* requires early years’ teachers to demonstrate an understanding of multiplication. They should recognise the importance of the array structure over the equal group structure that encourages additive thinking when supporting children’s transition to multiplicative thinking (e.g., Anghileri, 1989). The array structure provides conceptual understand of multiplication for children and commutativity, also the basis for generalising the factor, factor, product idea which is important to support fraction representation (e.g. Hurst & Hurrell, 2016).

When considering PSTs’ justifications we were not surprised at the concentration of mathematical ideas being attended to for samples B and C because they provide evidence of students at Level 2, in that they drew a response with an array and /or groups and explained their reasoning in words and/or with symbols and numbers (ACARA, 2016). It could be argued that Sample C shows the student is working toward Year 3 because they have recorded multiplication facts as part of their response.

In summary, the PSTs tended to focus on children’s strategies, and some mathematical ideas related to multiplicative thinking and student misconceptions when selecting and sequencing children’s work samples to share during a mathematical lesson.

The PSTs indicated that this was a worthwhile task as illustrated by the following PST reflection:

Yes - because it allows us [PSTs] to have a better understanding of what children are learning and what to look for.

## Report on Pre-service Teachers’ Descriptions within an Assessment Rubric

The second cohort responded to the rubric activity after completing the selecting and sequencing activity. As indicated earlier (method section) the codes (see, Table 4) were used to gain an overall sense of each pair of PSTs’ rubric description and evidence of their foundation knowledge and connection using further coding from the Knowledge Quartet (Rowland et al., 2009). The following are two examples of four PSTs’ responses to the rubric activity. Figure 7 provides an example of foundation knowledge and/or making connections in the descriptors for *content* and *process*.

|  |  |  |
| --- | --- | --- |
| Content Code | ../../../../../Desktop/Screen%20Shot%202017-03-27%20at%2012.20.1 | Content Code |
| ii | i |
| i | ii |
| i | i |
| i | i |
| i | i |

*Figure 7.* PSTs’ summary description for rubric activity and codes (Pair 3).

Figure 7 shows Pair 3 responses reflect generic descriptions (i) with little evidence of mathematical content related to the array task and evidence of foundation knowledge. There is, however, evidence of sequencing across Scores 1 to 5. As these PSTs demonstrated limited MCK, they are likely to have difficulties when making connections to the student’s next stage of learning and multiplicative understanding.

Figure 8 is an example of a pair of PSTs who demonstrated evidence of foundation knowledge and/or making connections when generating their descriptors for *content* and *process*.

|  |  |  |
| --- | --- | --- |
| Content Code | /Users/adownton/Desktop/Screen Shot 2017-04-07 at 1.53.23 PM.png | Content Code |
| iii | iii |
| iii | iv |
| iii | iii |
| iii | iii |
| iv | iii |

*Figure 8.* PSTs’ summary description for rubric activity and codes (Pair 5).

Figure 8 shows Pair 5 have relied on choice of mathematical language related to multiplicative concepts throughout the rubric. They demonstrate well developed foundation knowledge, needed and used for reporting children’s mathematical growth and knowledge. Such understanding then can be used when considering connections, such as planning for learning and/or assisting children who have difficulties or demonstrate misconceptions.

Having constructing their rubrics each pair of PSTs (n=20) assessed the five children’s responses to the array task using their rubric. Most (8 pairs) assigned samples C and B as task accomplished (Score 4) as they considered these work samples reflected the key mathematical ideas underpinning the task. One pair of PSTs allocated Sample C as Score 5 because in their opinion, the children “used more technical thinking.” Possibly they were unable to consider knowledge at the horizon for their assessment of children’s understanding, due to their beginning knowledge of children’s early multiplicative thinking. Eight pairs of PSTs assigned Sample D as substantial progress Score 3 as it reflected some understanding of the array structure. However, they tended to overlook the incorrect calculation of the multiplication fact and the unequal distribution of the chocolates in the array. Most PSTs agreed that sample A was an example of some progress (Score 2) and Sample E, little progress (Score 1).

These results align with the sampling and sequencing activity in which most PSTs chose samples B and C as the ‘strongest’ or evidence of task accomplished. They also align with the results in Table 6 that shows most PSTs’ justifications related to children’s *strategies* and *mathematical ideas.* These PSTs were also able to demonstrate developing, *overt subject knowledge* and *concentration on procedures* (see, Table 3).

The actual student work samples also provided a challenge for some PSTs when considering the processes (column 2) because they had difficulty interpreting the students’ recordings. For example, the fact that the student did not draw an array (Sample A) may have prompted the PSTs to suggest it was incorrect, rather than considering the child’s thinking when solving the problem. Overall the PSTs’ justifications indicated their beginning knowledge for identifying strengths and limitations of student mathematical thinking evident in their rubric descriptions.

The PSTs’ reflections, post completion of the rubrics, were informative in so far as they confirmed some of our findings relating to the lack of depth within the rubric and their limited *foundation knowledge*. Many indicated that differentiating the language when defining each category of the unit was challenging, as was interpreting the work samples, and understanding the mathematics sequence. Some of PSTs responses included:

What the children’s samples showed and what strategies they are actually using

Ensuring there was a significant variation between each score criteria, e.g., progressively increasing the requirements of the task in terms of multiplication concepts.

Differentiating between content and processes

Basic level of understanding mathematical sequences

Anecdotally several indicated it was a worthwhile task although very challenging, and provided them with a sense of what to expect when on their practicum. Two PSTs also wrote:

…it made me understand how difficult it can be and to assess children and their knowledge…

… it was really helpful in establishing my understanding of assessment and the elements of the multiplicative topic to be assessed…

In summary, these findings suggest that the PSTs’ rubric content descriptions were not task specific and had limited connection to mathematical language and/or concepts of multiplicative thinking or making connections to the children’s work samples. Further follow up could include the importance of how and what to interpret rather than generalise what they think the child might know. The task therefore has potential to purposefully develop PSTs’ *foundation knowledge.* Similarly, having the PSTs first complete the task as learners may have enabled them to generate the rubric descriptions with greater understanding of the mathematics underpinning the task. Another possible limitation was providing the PSTs with only five student work samples rather than those of the entire class. Doing so would offer further insight into the range of student understanding evident within any one class and encourage development of their *connected* knowledge.

## Reports on Foundation Knowledge and Connections within Rubric Descriptions

To situate the results with reference to the Knowledge Quartet the following discussion focuses on five codes (see method section) associated with the two dimensions of *foundation* *knowledge* and *connection* (Rowland et al., 2009). When considering evidence of *anticipation of complexity*, the pre-service teachers were making links between their descriptions for Score 1 to 5. In relation to evidence of complexity the PSTs rubric content descriptions lacked the detail of recording mathematical evidence contained within the children’s samples. For example, Score 1 (little progress) could indicate some attempt to draw an array, however in Sample E (Figure 6) the child did not draw an array with equal rows (see example Appendix B, Sample A, Score 1). Similarly, for Score 4, task accomplished would suggest the child can visualise the array structure and draw an array that reflects the size of the box within the problem. Such as, four rows of six or four rows of ten as recorded in Samples A and B. Recording may include an explanation of thinking and a correct number sentence and/or drawing of a suitable array (see, Figure 2).

When planning and making decisions about sequencing lessons teachers make links to previous lessons and links within lessons by relying on their foundation knowledge (Rowland et al., 2009). When considering the children’s responses to the array task, PSTs relied on their MCK when deciding the sequencing of their rubric descriptions. There was some blurring of PSTs’ understanding of the difference between content versus process, possibly unfamiliar terminology for many.

Making connections involves the teacher being able to anticipate the complexity of the task and breaking an idea down into steps that can be understood by students (Rowland et al., 2009). Making connections was evident when PSTs noticed a misconception, such as in Sample E, and be able to consider how to assist the child to further develop their understanding of arrays. For example, providing counters to model the situation or to provide an empty chocolate box that has arrays. Making connections between concepts links with Ma’s (1999) description of breadth, “the capacity to connect a topic with topics of similar or less conceptual power (p. 124).”

Further analysis of each PSTs’ responses and rubric data would explore *breadth* of knowledge and making connections between concepts. Rowland et al., (2009) suggested when demonstrating *foundation knowledge,* the teacher concentrates on developing understanding rather than relying on procedures. Our study and that of Meikle’s (2016), highlight the importance of PSTs being able to demonstrate their *horizon content knowledge*. In our study, PSTs had difficulty describing ‘goes beyond’ and in Meikle’s study she identified that middle year PSTs had difficulty providing a rationale beyond mathematical procedures.

Furthermore, a justification for engaging PSTs (or teachers) in designing a task specific rubric assists them to extend their *foundation knowledge* and make *connections* with children’s mathematical understanding. Second with the increased use of open ended tasks teachers can assess whilst children are learning. Yeo, (2011) found that early years teachers need to develop the skills and confidence to implement alternative assessment practices which includes the use of open ended tasks and develop task specific rubrics. Providing PSTs with experiences such as generating a task specific rubric can assist their development as early years mathematics teachers.

Conclusion and Implications

In this study, we have endeavoured to link theory and practice through engaging the PSTs in activities that related directly to how children learn mathematics in the early years. The PSTs were able to clarify and justify the choices they made when analysing students’ work samples to a challenging task related to multiplication and arrays. We also drew on the Knowledge Quartet (Rowland et al., 2009) and work of Smith and Stein (2011) to gain specific insights into PSTs’ *foundation knowledge* and *connection* related to multiplicative thinking. With respect to the research questions the findings indicated the following conclusions:

First, these PSTs required further *breadth* and *depth* of knowledge when making connections between the process and concepts underpinning multiplicative thinking.

Second, well-developed *foundation knowledge*, including knowledge of the stages children move though in their transition from additive to multiplicative thinking is essential when considering children’s work samples while *selecting and sequencing* during the summary phase of the lesson and when designing a task specific assessment rubric.

Third, having PSTs design a task specific rubric highlighted their difficulties when identifying key concepts of multiplicative thinking, and the complexities involved in articulating and interpreting children’s work samples.

Fourth, a key component of formative assessment is having the skills to analyse and interpret children’s work samples and developing coherent and concise statements of understanding within a rubric is complex and required PSTs to use specialised teacher knowledge.

Fifth, engaging PSTs’ in practice-based activities such as those described in this paper were effective ways to link theory and practice; provides guidance for future practice, and teacher educators with insights into gaps in PSTs *foundation knowledge* which might otherwise have gone undetected. As Grossman et al. (2009) indicated, representations of practice, such as children’s work samples, have the potential to support PSTs when learning, not just about teaching, but how to use knowledge of teaching in action.

In light of these conclusions, when reviewing our course content, we need to ensure PSTs have the opportunity to extend their *foundation knowledge*, including conceptual knowledge of children’s misconceptions when understanding the transition for additive to multiplicative thinking. Second, given assessment is a priority and important for assisting children’s growth in mathematical understanding, we need to spend time assisting PSTs’ development of their formative assessment skills. We argue that the activities PSTs engaged in within this study were innovative and supported knowledge for teaching mathematics in the early years relating to the use of challenging tasks, selecting and sequencing, and assessment practices.

# References

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## Authors

First author's name

First author's institutional address

email: First author's email address

Second author's name

Second author's institutional address

email: Second author's email address

Third author's name

Third author's institutional address

email: Third author's email address