Evidence of Teacher Change after Participating in TRIAD's Learning Trajectories-based Professional Development and after Implementing Learning Trajectory-based Mathematics Instruction[1]

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Increased attention has been given to learning trajectories (LT) as structural frameworks for educational instruction. The purpose of this study was to explore preschool teachers' descriptions of self-change, seven years after the start of their participation in LT-based professional development and instruction. This study was part of a larger study to test the effects of a model for scaling up educational interventions also based on LT. We used grounded theory methodology to study teachers' descriptions of self-change. The dominant change was teachers' newly acquired awareness of the mathematical capabilities of preschoolers. Teachers' ability to notice these capabilities was scaffolded by their knowledge and use of LT in implementing the mathematics curriculum, but change occurred as the teachers noticed, listened to, and elicited talk from the children. In brief, teachers learned from the children. Teachers appeared to be on their own trajectories of learning, as they began to build mental models of their children's mathematical thinking. Together, these trajectories may provide a theory of teacher learning.

Keywords: learning trajectories · in-service teacher growth · child as teacher · teacher as learner · teacher learning trajectory

Introduction

There is a need for high-quality early childhood mathematics teacher education programs, especially as research studies provide robust evidence that young children's mathematics skills predict not only later mathematics achievement but also later school success (Claessens & Engel, 2013; Geary, Hoard, Nugent, Bailey, 2013; Jordan, Kaplan, Ramineni, and Locuniak, 2009; Watts, Duncan, Siegler, & Davis-Kean, 2014). Further, although young children are ready and eager to learn mathematics (National Research Council, 2001), in the U.S. most young have limited opportunities to learn mathematics, especially because many early childhood teachers are not well-prepared to engage children in rich experiences in domains other than literacy (Institute of

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Medicine (IOM) and National Research Council (NRC), 2015; National Research Council, 2009). Given the limitations in the present preparation of the early childhood workforce in the domain of mathematics education (Institute of Medicine (IOM) and National Research Council (NRC), 2015), scaling up professional development has special challenges in preschool contexts, including the diverse workforce, the equally diverse knowledge of teachers, and the low level of mathematics content and pedagogical content knowledge of most preschool teachers (Sarama & DiBiase, 2004).

The connection between early mathematics skills and later academic success, and the limited training most early childhood teachers receive in mathematics, increase the importance of designing effective programs for teacher education and in-service professional development (PD) for early childhood teachers. Researchers have focused on ascertaining the essential features of high quality professional development (e.g., Björklund & Barendregt, 2016; Brendefur, Strother, Thiede, Lane, & Surges-Prokop, 2013; Carpenter, Fennema, Franke, Levi, & Empson, 2014; MacDonald, Davies, Dockett, & Perry, 2012; Perry, 2010; Piasta, Logan, Pelatti, Capps, & Petrill, 2015; Sarama & Clements, 2017; Sarama, Clements, Wolfe, & Spitler, 2016), leading to evolved understanding and growing consensus on the most important features of effective PD.

Learning Trajectories-based Professional Development (LTPD)

Many researchers concerned with pre-service teachers and in-service professional development emphasize a need for teachers to understand children's mathematical thinking (Baroody & Ginsburg, 1990; Carpenter, Fennema, & Franke, 1996; Franke, Carpenter, Levi, & Fennema, 2001) and some have documented the connection between teachers' understanding of children's mathematical thinking and achievement (e.g., Carpenter, Franke, Jacobs, Fennema, & Empson, 1998). A growing body of evidence shows that children's thinking does not always follow historical or logical developments of mathematics as a discipline, nor adult ways of reasoning (e.g., Björklund & Barendregt, 2016; Philipp, 2008; Sztajn, Confrey, Wilson, Edgington, 2012).

The construct of learning trajectories (LT) (Simon, 1995) further stimulated focus on the natural sequences of children's mathematical growth (Clements & Sarama, 2004; Confrey, Maloney, & Corley, 2014). LT, closely related to learning progressions (Corcoran, Mosher, & Rogat, 2009) and learning-teaching trajectories (van den Heuvel-Panhuizen, 2008), have drawn the interest of researchers in an effort to sequence the progression of children's thinking, and to study the implications for education. This interest has generated empirical documentation of the sequences of children's thinking in several subject areas, including mathematics (Battista, 2011; Bobis et al., 2005; Clarke, 2008; Clements & Sarama, 2009; Confrey, Maloney, Nguyen, Mojica, & Myers, 2009).

Learning trajectories are more complex than sequencing of topics or tasks, and for the purpose of this paper, we define mathematical learning trajectories as:

> descriptions of children's thinking and learning in a specific mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain. (Clements and Sarama, 2004, p. 83)

The inclusion of the instructional components separates this definition of learning trajectories from "progressions" or other research that only documents natural sequences of children's thinking. All aspects of the present study, especially the professional development, were based on this definition.

Supporting this LTPD approach, Sztajn, et al., (2012) argued that learning trajectories-based instruction (LTBI) served as a tool for teacher education by providing a framework for organizing theory on teaching and learning. Supporting this view, a LT on equi-partitioning was shown to support teachers' understanding of mathematics, as well as their students' reasoning (Wilson, Mojica, & Confrey, 2013). Such work has generated attention to learning trajectories (LT) as a structural framework for students' education (Sawyer, 2014). Importantly, learning trajectories play a central role in the AMTE Standards for Mathematics Teacher Preparation (Association of Mathematics Teacher Educators, 2017), especially for early childhood.

A LTPD approach is consistent with, and we believe can support, professional development that considers teachers' classrooms as professional development laboratories (Cobb, Wood, & Yackel, 1990), wherein teachers learn to learn from the child. After careful noticing, listening, and eliciting, teachers may subtly revise the known learning trajectory for a preschool mathematical domain, engage in local instruction (Gravemeijer, 2004) and create an instructional task that more exactly suits a particular student's needs, becoming "teachers as researchers" (Franke et al., 2001) who notice, professionally, levels of the trajectories not yet recognized and documented (Chinn & Sherin, 2014). Such teachers may recognize that they are teacher learners (Bobis et al., 2005), on their way to becoming lifelong professional learners (Feiman-Nemser, 2001). They may continue to seek and participate in external PD, recognizing that professional development matters (Darling-Hammond, Wei, Andree, Richardson, & Orphanos, 2009); mathematical knowledge for teaching (MKT) matters (Ball, Thames, & Phelps, 2008), training in noticing matters (Jacobs, Lamb, & Philipp, 2010), and training in learning trajectories matters (Clements, Sarama, Spitler, Lange, & Wolfe, 2011). Just as LTBI may provide a theory for teaching (Sztajn et al., 2012), such data-driven theoretical inquiry points toward LTPD as a possible theory for teacher learning and teacher change, with the child at the centre.

Indirect, and perhaps inadvertent, support for this approach can be found in some of the graphic models of professional development theoretical frameworks. Although some PD models display unidirectional arrows representing input from teacher to child, several models do depict reciprocal impacts between student and teacher. These models display arrows pointing from "change in student learning outcomes" to "change in teachers' beliefs & attitude;" (Guskey, 2002); from "salient outcomes" in students to "teacher knowledge, beliefs, and attitudes" (Clarke & Hollingsworth, 2002); from "improved student learning" to "change in instruction to increased teacher knowledge and skills; change in attitudes and beliefs" (Desimone, 2009); from "student needs" to "professional inquiry" to "[teacher] knowledge & skills" (Timperley, 2011); from "child achievement" to "sustenance – persistence of effects and continuation of intervention" (Sarama, Clements, Wolfe, & Spitler, 2012), and from "student behaviours" to "teachers' decisions" (Sarama et al., 2016). The professional development model depicted by Philipp (2008) puts children in the centre, with arrows from the children emanating outward.

The Context

The current study is part of a large cluster randomized trial designed to evaluate the effectiveness of a research-based model for scaling up educational intervention that put learning trajectories at the core. Effects on children and teachers were positive in the short and long terms (explication is in the following Methods section). However, we had not studied how teachers perceived changes to their beliefs, views, and practices, information that will better inform the future development and application of LTPD and LTBI.

The Present Study

Thus, our research question was: What can be discovered about the effects of teacher participation in learning trajectories-based professional development and learning trajectories based curriculum implementation from teachers' written descriptions of self-change? To answer it, we engaged in a grounded-theory analysis of the written descriptions of teachers' perceptions of personal change experienced during and after their participation. We conducted a theoretical inquiry of written self-reported data obtained from teachers who responded to open-ended questions about the ways in which they changed as a result of participation in a learning trajectories-based professional development and instructional program.

Methods

The TRIAD Model and Evaluation

The Technology-enhanced, Research-based, Instruction, Assessment, and professional Development (TRIAD) model emphasizes teaching early mathematics for understanding via learning trajectories and technology and includes collaboration of key groups to establish and maintain (a) a pre-K mathematics curriculum, with all components of the curriculum—a teacher's manual, demonstration videotapes, manipulatives, software, teaching strategies, assessments, and professional development—based on a common core of understanding the learning trajectories through which children develop mathematically; (b) professional development for teachers; (c) on-site support for teacher by facilitators during the school year; and (d) supportive roles and materials for parents. These are the comprehensive components necessary for improving teaching (Association of Mathematics Teacher Educators, 2017; Hiebert & Stigler, 2017). TRIAD's collaboration with key groups is dedicated to developing organizational structures that intensify and focus, rather than dissipate and scatter, teachers' motivation to engage in and maintain this challenging practice (for a full explication of TRIAD's 10 research-based guidelines, see Sarama & Clements, 2013; Sarama, Clements, Starkey, Klein, & Wakeley, 2008).

The TRIAD evaluation tested for generalizability of the model over two diverse settings in two states in the U.S. Although the Common Core State Standards (Council of The Chief State School Officers (CCSSO) and National Governors Association (NGA), 2010) do not include preschool, both school districts in which we worked followed state mathematical standards for this age. We employed hierarchical linear modelling (HLM) to measure the effects of the intervention on children's mathematics performance (Clements et al., 2011; Sarama et al., 2012; Clements, Sarama, Wolfe, & Spitler, 2013, 2015). These studies revealed that teachers implemented the curriculum with good fidelity, increasing children's mathematical competencies (effect size = .72) as well as showing transfer to other domains, such as oral language (Clements, Sarama, Layzer, Unlu, Germeroth, et al., 2017; Clements et al., 2011; Clements et al., 2013; Sarama et al., 2008; Sarama, Lange, Clements, & Wolfe, 2012). We also tested for the sustainability of teachers' implementation. Not only did fidelity of teaching with learning trajectories not decrease, as expected, after project support was discontinued for two years, it increased (Clements et al., 2015). Recently, we found these teachers evinced yet higher quality of mathematics teaching five years after support for the project had ceased (Sarama, Clements, Wolfe, & Spitler, 2016). However, greater understanding of the processes of change experienced by teachers, who play a key role in TRIAD, was needed.

Participants

Original participants were 106 preschool teachers from 42 schools. Schools were randomly assigned, by site, to one of three treatment groups: two were experimental (72 teachers in 26 schools), with no differences relevant to this study (differences in the experimental treatments began after the preschool year) and are referred to together as such here, and one was the control

(34 teachers in 16 schools). Schools served low-resource, ethnically diverse children in two districts, each in a different state in the U.S. Every teacher in each school was required by their administration to participate and was assigned to an experimental or control group based on the school assignment. Teachers were required to attend all PD sessions (experimental teachers for 13 full days over two academic years, mostly on school days; control teachers for approximately 8 days over one academic year, on a delayed basis, after child outcome data were collected).

Data for this study are from 73% of the original teachers (77/106), representing 83% of the originally randomized schools (35/42), seven years after the start of experiment. For purposes of this study, there was only group: 77 teachers, both experimental and control, all of whom participated in LTPD, implemented the learning trajectories-based curriculum, and voluntarily participated in a teacher questionnaire, seven years after the start of the study. Some of the 77 teachers (53) participated in two years of LTPD and some (24) in one year of LTPD. Each teacher's responses were compared to all other individual responses, rather than comparison of responses by teacher groups. Demographically, 97% had a master's degree; 68% were state certified; all but one were female. On average, they had 21 years of teaching experience and an approximate average of 12 years of experience teaching at the preschool level.

Original children (1,375) of teacher participants were mostly four-year-olds (51% Female) of mixed ethnicity (53% African American, 21% Hispanic/Latino, 19% White, 3.7% Asian Pacific, 1.8% Native American, and 0.01% Other). Most (82%) received free or reduced lunch, 13.5% had limited English proficiency, and 10% had an Individualized Educational Plan for special services. District mandated class size ranged from 18 at Site 1 to 24 at Site 2. Although child outcome data were collected on 15 children randomly selected from each class (average number of participants was 13.6 per classroom), all children from each classroom participated in the implementation of the intervention.

Intervention

Learning trajectories were the core of the professional development program. Teachers, in cohorts ranging from 20 to 24, actively participated in LTPD for 8 days during Year 1, and 5 days during Year 2, (control teachers for approximately 8 days during Year 3) at off-site locations in each city. The LTPD sessions, held approximately monthly and mostly during the school day, were paced and organized to follow the sequence of a 30-week early mathematics curriculum. This description focuses on the teachers' experience with the LTPD sessions.

During the first half of each session, teachers actively learned about the three components of the learning trajectories (mathematical goals, children's developmental progressions, and instructional tasks). Presentations on mathematical concepts and ideas facilitated teachers' understanding of mathematical end goals (cf. Polly et al., 2015; Stosich, 2016). Teachers learned about the second element of the trajectories, the development progressions, by viewing and discussing multiple videos of preschoolers' thinking out loud mathematically as they interacted with an adult around mathematical materials and problems. Individual videos illustrated particular levels on the pathways, serving as landmarks for teachers to identify progress, and as reference points for assessment (van den Heuvel-Panhuizen, 2008). Finally, the instructional tasks, activities, and tools that facilitate movement along the path from informal ideas to more sophisticated understandings were demonstrated, discussed, and explored, setting the stage for the afternoon sessions. As a group, teachers also watched videos of other preschool teachers teaching the actual curriculum lessons to actual preschoolers in diverse settings, observing varied instantiations or adaptations of these lessons within the parameters of the lesson objectives.

During the second half of the LTPD day, teachers practiced. Teachers gathered for "whole group" activities in a simulated setting, enacting the whole group lessons by taking turns leading the lesson, and/or playing the part of preschoolers. They practiced "small group" by gathering in groups of four to simulate the small group sessions they would lead in their classrooms. Early in the year, project staff demonstrated instruction. Gradually, teachers took turns leading the lessons while others posed as preschoolers. A summary of the learning trajectories was available for teachers for reference.

Formative assessment of children's place on the continuum of the learning progressions was an expected part of the small group implementation. Teachers became familiar with the "small group record sheets" they would use to formatively assess every student during every small group lesson they taught in their classrooms. This became routine for the teachers over time. They became skilled at using the nouns the developers chose to represent the learner at every step along every learning progression (e.g., for counting beyond verbal counting, "corresponder" or "producer"). These terms facilitated quick and accurate formative assessments. Teachers discussed how to observe, listen to, and interpret children's thinking. They learned that formative assessment is a powerful tool for decision-making, for knowing what to teach to whom and when. In particular, developmental levels provided teachers with a good range of levels to utilize when working with children who were gifted and/or had special needs.

At the computer, teachers independently practiced the curricular computer activities they would introduce to the children in the upcoming weeks. Teachers also visited the LTPD website where they could view videos of children at specific steps of the trajectories with descriptions and commentaries (e.g., see LearningTrajectories.org, a recent adaptation of this web-based tool). Observing videos of the lessons provided opportunities to see connections between the learning trajectory's level of thinking and the specific instructional activities. Additionally, teachers could interact with the data management system that tracked the progress of each of their children on every software activity. Tracking children's progress allowed teachers to tailor assigned software activities to the level of both individuals (i.e., children who needed either earlier or more challenging activities) and/or the whole class at the beginning of each week.

Overall, teachers experienced, in a collegial atmosphere, structures simulating what they would provide for their children; they were in communities of learners (Penuel, Fishman, Yamaguchi, & Gallagher, 2007), acting around boundary objects (Sztajn, Wilson, Edgington, Myers, & Teachers, 2014). By increasing their knowledge of mathematical activities to provide for children based on children's positions on the mathematical trajectories, teachers increased their understanding of children's mathematical thinking. The integration of the LT into every aspect of professional development sessions offered coherence as the instructional tasks connected to both the developmental progressions and the formative assessments. Teachers could then make sense of the LTPD sessions, and simultaneously gain confidence in their ability to implement and formatively assess their children.

During Year 2, teachers focused less on the curricular activities, and more on the mathematical learning trajectories for preschoolers, presenting case studies to their peers. Teachers discussed the cases, assessed where the child was on a developmental progression, and discussed what activity, or modification of an activity, might be appropriate for the particular child.

LTPD also took place in the teachers' own classrooms via the LTPD website and visits from project coaches and peer coaches. Project coaches were TRIAD staff who received two days of training on TRIAD coaching. They participated in all LTPD sessions, and provided individualized support during these sessions. Project coaches interacted with their assigned teachers (four to five teachers per coach) approximately twice per month. At times the teacher would observe as the project coach demonstrated small or whole group, or software activities, but usually the project coach would observe and support while the teacher implemented the curricular activities. Project coaches completed fidelity ratings 3-4 times per year, provided teachers with feedback, and offered teachers a fidelity rating sheet teachers could voluntarily use to self-evaluate.

School-based peer coaches were teacher participants who volunteered to lead all participating teachers at their school. They served as a liaison to the project staff and provided in-school/in-classroom support for the teachers as they implemented the intervention. Technology mentors (project staff) rotated into classrooms to troubleshoot software, hardware, or network problems, demonstrate software, and answer teachers' technology questions. Website activities allowed teachers to test themselves by observing a video of a child and naming the child's location on a progression, using the descriptors for each level.

Mode of Inquiry

The analytical method was a theoretical inquiry into the written responses of 77 teachers who were asked, via seven open-ended questions, to describe how their beliefs, views, and practices changed as a result of participation in professional development and curriculum implementation based on the learning trajectories. We used grounded theory (Charmaz, 2014; Thornberg, 2012) processes of coding, focused coding, categorization, abstract coding, constant comparison, theoretical memoing, theoretical sampling of the data in relation to categories, theoretical saturation of categories via sampling of data, and data driven theory development (Charmaz, 2014; Hood, 2010). We searched for and identified negative cases that contradicted the major pattern (Charmaz, 2014), and integrated them into the ongoing analysis. As the data set was limited in that the data sources were written responses to a survey and we were unable to return to the participants to sample emerging theory, the use of theoretical sampling of the literature in addition to data suited the study (Lo, 2016; Thornberg, 2012).

Data Sources

To study teacher change as a result of participation in learning trajectories-based professional development (LTPD) and curriculum implementation, we invited all 106 original teachers to fill out a paper and pen Teacher Questionnaire seven years after they completed the original Teacher Questionnaire (TQ). The later TQ (sent to teachers' homes via mail in 2012) included the same items as the first, as well as additional questions, many on the topic of teacher change. Most of these additional items were 4-point Likert style questions; twelve items required written answers. This study examines teachers' self-reported responses, seven years after the start of experiment, to the seven of these twelve questions that involved teacher change.

We did not presume change. The questions that preceded six of these seven questions were 4-point Likert style questions that set up a condition. Example: *The TRIAD/Building Blocks professional development sessions and the TRIAD/Building Blocks program changed how I view students and how I taught them as learners of mathematics. 1 Strongly Disagree; 2 Disagree; 3 Agree; 4 Strongly Disagree.* Teacher responses to these questions are not reviewed here, but were considered in the analyses. If a teacher denied change, we did not expect a response to the question. The following item illustrates the format and wording of the follow-up, qualitative question: *If you agreed with the above statement, please describe how your views/practices changed.* Three blank lines followed each question. A similar question appeared later in the TQ: *In what ways, if any, have your beliefs changed?* Five questions asked about how teachers changed their views/practices in relation to particular groups of children. Again, these questions were preceded by the conditional question as to whether or not there was change. Example: *The TRIAD/Building Blocks professional development and overall program changed how I view girls as learners of mathematics. 1 Strongly Disagree; 2 Disagree; 3 Agree; 4 Strongly Disagree,* followed by, *If you agreed with the previous statement, please describe how* *your views/practices changed.* The other groups included special needs, gifted, Hispanic, and African American children. Again, these questions were followed by three blank lines.

Teachers typically gave one response to each question, usually in a sentence or two. (Many left blanks in questions about ethnicity and special groups. The average length of the responses to all seven questions was 18.3 words. The shortest was one word (there were 6 of these: 1 none, 1 same, 4 N/A); the longest response was 69 words. For clarity of presentation, rather than presenting responses by question number, we organized the data by topic/category/theme, and integrated teachers' responses on changes in their views/practices and beliefs, following Franke et al., (2001), who posited that "because beliefs and practice are interrelated in complex ways, it is more productive to integrate beliefs and practice into a single scheme" (p. 661); and Cobb et al., (1990), who argued that "when analysing the… teachers' learning, her beliefs and practices were interdependent and developed together" (p. 145).

Results

Preschoolers' Mathematical Capabilities

The most dominant responses to questions on teacher change involved teachers' newfound awareness of their preschoolers' mathematical capabilities (66% of the teachers wrote about children's capabilities). Teachers' responses fell into five groupings: preschoolers are more capable than I thought; I have raised my expectations of preschoolers' mathematical capabilities; they can learn complex mathematical ideas; they can learn a variety of mathematical concepts/topics. The teachers also wrote explicitly about what the children could do mathematically.

Capabilities. The following examples are representative changes in teachers' knowledge of children's capabilities: "Children are working with mathematical concepts and vocabulary I had thought they would never be able to understand"; "I realize how much more I can accomplish and teach to 3, 4, and 5 year olds." Teachers frequently mentioned the word "capable" in their responses: "Pre-K children are capable of doing and learning more concepts than I thought." Others answered similarly: "I know what children are capable of because I have seen it;" "[T]he students indeed could do what we think they just can't;" "Children are able to learn more than what we think;" "Some of these skills and concepts, I thought, that preschool children were not ready to comprehend;" and "Four-year-olds can do more than you think with these math concepts."

Higher expectations. Some teachers wrote about raising their expectations after becoming familiar with their preschoolers' mathematical capabilities. One teacher responded: "I now believe in raising the learning bar very high with young learners and giving them support to come to mathematics reasoning and experiences on 'their own'." Another wrote: "Young children can meet and master higher expectations and concepts in math than I had previously thought." And another answered: "TRIAD/Building Blocks changed how I view students because now I feel that nothing is too hard for them to learn." Again, in the vein of raising expectations for the preschoolers, one teacher responded: "My beliefs have just expanded [to a] further/higher level for my 4-year-olds." Similar responses include: "My standards and beliefs of what 4-year-olds are capable of have dramatically changed;" "My standards are higher;" and "I now believe that students, even this young, can understand, and engage in much higher-level math than before."

More complexity. Two teachers described their changed beliefs on the complexity and the difficulty of the mathematical ideas the children were able to work with: "Children are much more capable of learning complex ideas;" and "[I] presented students with …more difficult concepts." One teacher described her amazement: "I was amazed by the degree of difficulty of the lessons (concept) but the way we teach them, four-year-olds 'get it'!!" In describing what changed, one teacher with one year of PD wrote: "My beliefs in what concepts children this age are ready and able to understand;" Another wrote: "At first I thought this is too much for pre-k until I started implementing the practices into my classroom and saw just how ready they were for it and how it was easier than I thought to use it in my classroom."

More variety. Teachers also expressed surprise at the variety of topics children were able to learn. "They are able to understand/connect to more topics than I thought previously;" "I just realize my students are more capable of learning various math concepts I hadn't taught before;" "[I changed in that I now understand] the variety of mathematical concepts children are capable of understanding at age three and four-years-old;" "I presented students with more variety in math learning."

What the children can do specifically. When asked to describe how their views/beliefs/practices had changed as a result of their participation in the LTPD and LTBI, most teachers responded by acknowledging their new awareness of their preschoolers' mathematical capabilities. Instead of asking teachers what they found impressive about their preschoolers' capabilities, we asked them open-ended questions about self-change. Yet, it appeared that many teachers, 44%, observed surprising mathematical abilities in their children, which contributed to remarks made about preschoolers' capabilities. We present one representative statement here, and then a summary using the teachers' words of what they noticed about their preschoolers' capabilities: "I now know that children who are 3, 4, 5, years of age are able to explain or justify the strategies they used to solve problems that involve mathematics." Comments below from 34 individual teachers are grouped into four categories. Each unit ('meaning unit') is discrete and coded uniquely.

1. *Reason, explicate, justify, and problem solve.* Preschoolers can verbalize their reasoning; talk about how they came up with the correct answer, or why; reason, think and solve many problems; can provide a reason; problem solve and explain their thinking; explain their mathematical thinking; justify their thinking; prove it; explain or justify the strategies they used to solve problems that involve mathematics; justify their strategies; express themselves; share ideas.
2. *Lead the way.* Preschoolers can lead; excel; flourish; accomplish more; become smart and fast; expand their knowledge; do more than you think with these mathematics concepts.
3. *Understand complex ideas, make connections and understand relationships.* Preschoolers can understand more difficult concepts; understand difficult lessons; learn complex ideas; meet and master higher concepts in mathematics; understand and engage in much higher-level math; understand/connect to more topics; use real materials to understand mathematical concepts and relationships.
4. *Specific groups of children*. Five of the seven questions on teacher change involved changes in teachers' views/practices/beliefs about teaching specific groups of children mathematics after participating in LTPD and LTBI. Many teachers left these questions blank. Some teachers indicated that there were no changes in the ways they taught mathematics to children who belonged to these groups (girls, special education students, gifted students, Hispanics, and African Americans). Examples of these follow: "Believing that girls can be learners of mathematics equally as boys, my view never changed about that. Being a woman I always believed that girls were equal to the boys mathematically". Others wrote: "I don't feel race has anything to do with math"; and "I like to view all my students by who they are as people, not by their colour. This program did not change my view at all". Some teachers described self-change in relation to these groups of children. One teacher wrote: "I had a mild bias on how girls learned mathematics. My changes include a less biased approach and new examples that can be related to by both boys and girls". Another wrote: "I call on girls more". Regarding special education, some wrote:

"The TRIAD/Building Blocks program has made me more aware of teaching strategies I can apply for all level learners"; "The professional development helped me plan for children and differentiate learning according to their individual special needs"; and, "The program help[ed] me view special needs students in a clearer way". Self-changes in regard to gifted students were described: "I had new strategies to use to encourage gifted learners to extend their learning and applications"; "The gifted students really excelled… I was actually able to identify gifted students sooner". Of those who described change regarding teaching mathematics to Hispanic students, one teacher wrote: "It helped to refocus my thoughts and gave me options to help my Hispanic students become more successful". Teachers who described self-change regarding African Americans wrote: "I saw more African American students excel in mathematics using the TRIAD/Building Blocks program" and "I call on them more and encourage their output".

Learning Trajectories

Teachers' responses to questions about teacher change after participating in LTPD and LTBI also centred on children's mathematical thinking and the mathematics learning trajectories for young children; the professional development sessions themselves; small group instruction and complementary supports to small groups (whole group, hands-on learning, and computer activities); formative assessment and differentiation; and teacher/child enthusiasm.

Learning trajectories and children's mathematical thinking. Eight teachers used the word "trajectories", specifically; for example: "The developmental trajectories are a very helpful analytic tool"; "The learning trajectories help me teach the lessons to various abilitied-level learners"; "I have a better knowledge base to teach from. I understand the trajectories to help me teach"; I have a better understanding of how to challenge them more – move them to the next learning trajectory [level]"; "I've made it a point to refer to the trajectories in order to understand the stages of learning of each learner". In writing about self-change in relation to learning trajectories, some used these words to describe their new knowledge: levels, steps, stages of understanding, developmental understanding, higher learning skill, step-by-step, sequential. In relation to children's mathematical thinking, some used phrases such as: "explaining how a child thinks" and "learning how children learn."

LTPD. Some teachers wrote about changes in relation to LTPD. They reported that they gained "a better appreciation for math"; "realized how important it is to teaching pre-k students"; became "more confident in teaching math to small children"; obtained "solid resources to develop my own math understandings"; gained "the skills, training and materials to be successful"; and became "empowered to give the children the best". Multiple teachers, in writing about self-change, wrote about practicing and rehearsing the lessons in PD. They mentioned the demonstrations, the explanations, the examples, the modelling, and the chance to practice each software activity in PD so that "the lessons are not abstract to us".

LTBI and small group instruction. In reporting self-changes, teachers described incorporating small group instruction and adding more small group instruction. One teacher indicated that "Small group learning helped in the sense that we can differentiate". Another wrote: "The small group, specific lessons showed me how much more the students can learn if taught this way". Another teacher indicated that "using methods, manipulatives, [and] small groups made these students and all students easier to reach, monitor and assess". Another attributed part of the program's success to the "daily assessments" that took place during small groups. One teacher stated that "I have included more small group and differentiated instruction to reach mastery of basic math skills". Some teachers described how they used components of the program to support their small group mathematics teaching by: introducing concepts in whole group; reinforcing through computer activities that complemented small-group lessons; using the various materials, activities, and computers to enhance the mathematics program; using the manipulatives to help children understand in an easy way; and using more hands-on materials.

Formative assessment and differentiation. Teachers said they learned to adjust groups and teach according to the needs of children; to differentiate learning according to individual needs; to allow children to come to their own way of doing mathematics that makes sense to them; to make daily assessments; to identify children who are having difficulty with skills by completing record sheets; to use the manual to differentiate skills; to adjust groups and adapt teaching methods; to watch children use manipulatives, and to listen to what they were saying rather than just teaching "at" them. One teacher wrote: "The children are able to express themselves better so that I can assess accordingly".

Teacher/child enthusiasm for a program suitable for children. In responding to questions on change, some teachers wrote about their own and the children's appreciation of the program. They used the following phrases to convey this change: realistic and do-able; children like the challenge; making it developmentally age appropriate as well as fun. One teacher wrote, "My beliefs have changed, because now we have a program where all students have the opportunity to excel in mathematics and have fun doing it" and another, "The children were more engaged and overall enthusiastic about math". One teacher wrote: "The math lessons I taught were more interesting and appropriate". Another wrote: "The system/program really helped engage students at their appropriate level".

No Change

Eight teachers indicated "no change," or "none" or "N/A" as a response to one or both of the two general qualitative questions about views/practices/beliefs. All eight teachers, in other responses on the same questionnaire, described positive changes.

Discussion

The growing body of knowledge about young children's mathematical capabilities (for reviews see Cross, Woods, & Schweingruber, 2009; Ginsburg, Lee, & Boyd, 2008; Mousley & Perry, 2009; Sarama & Clements, 2009) has not reached most early childhood teachers (Hachey, 2013). The preschool teachers in this study were experienced and well-educated; yet, when asked to describe self-changes after participating in Learning Trajectories-based Professional Development (LTPD) and Learning Trajectories-based Professional Instruction (LTBI), their most salient and dominant response was surprise at their children's mathematical capabilities. Two-thirds of the teachers focused on teachers' newfound understanding of preschoolers' mathematical capabilities; teachers' increased expectations for preschoolers' mathematical achievement; teachers' fresh recognition of preschoolers' ability to learn complex mathematical concepts; and teachers' new awareness of the variety of concepts/topics preschoolers could understand. Many of these teachers explicitly described their changed understanding of what their children could do mathematically, including the ability to reason, explicate, justify, and problem solve; lead the way; and understand complex ideas, make connections and understand relationships.

Other studies report that teachers have had similar experiences. During Sztajn and colleagues' study of teacher/researcher processes for exchanging knowledge around learning trajectories, teachers were asked "about the most significant learning from the encounter" (Sztajn et al., 2014, p. 2010). One teacher, when asked about the most significant idea she learned, responded: "not to underestimate kids and how much knowledge they bring to the experience—maybe that above all." Another teacher voice, from a study of classrooms as learning environments, said, "I have been teaching all this time, and I never knew second graders knew so much about math!" (Cobb et al., 1990, p. 134).

Another theme addressed changes in how they taught various groups (gender, ethnic/race, ability), always in the direction of equity and appreciation of all children's ability to learn mathematics.

It is interesting that when asked how they changed, teachers kept the focus on the children, explaining the change in their perceptions of children, and thus maintaining the child-centred focus of the LTPD and LTBI approaches.

In addition to the theme of children's mathematical capabilities, teachers, when asked about how they changed, described the effects of learning about (a) learning trajectories, (b) how to teach mathematics to young children, (c) how important small group instruction is in LTBI, (d) how to use formative assessment in early mathematics, (e) how to augment small group instruction, and (f) how early mathematics could constitute activities suitable for young children, incorporating mathematical activities through play. Although it appears that many components of the TRIAD implementation were influential and mutually reinforcing, teachers' responses do suggest that the changes they experienced occurred as they taught. That is, their newfound awareness of the mathematical capabilities of their pre-schoolers may have been catalysed by the LTPD sessions, but the fundamental changes seemed to take place as they interacted with, noticed, and listened to their preschoolers. That is, the change in teachers was scaffolded by the LTPD, but the children were also the teachers' teachers. Informed and prepared through participation in LTPD, these teachers were equipped, and, in one teacher's words, "empowered" to create the conditions in their classrooms for children's capabilities to evince themselves. The teachers practiced in PD, and they were ready to put the PD into practice (Anthony, Hunter, & Hunter, 2015; Lampert et al., 2013). They were empowered to notice and pinpoint children's locations on the learning trajectories. According to their self-reports, teachers implemented in their classrooms what they learned in PD (see also Clements et al., 2015 for results of confirming observational data). Teachers' TQ responses indicated that they set up small group instruction, the locus of discussion (Bray, 2011) and the major site for change in both children and teachers. That is, small group was the context in which instructional tasks were most profoundly enacted and experienced by both teachers and children; where preschoolers were explicit about their mathematical thinking; and where formative assessment and differentiation took place. During small group, teachers engaged children in LT based activities, attended to and noticed what they did (Jacobs et al., 2010), , listened to the children's explanations and reasoning (Empson & Jacobs, 2008), assessed (Heritage, 2008), located the children on mathematical learning trajectories, and differentiated. Teachers became increasingly exposed to the thinking and capabilities of children as they progressed through probable learning trajectories in several mathematical domains.

When questioned on change, many teachers explicitly mentioned the learning trajectories, describing self-change as they began to understand the components of learning trajectories. They also described the process of becoming familiar with the probable levels of thinking of preschoolers along developmental progressions, as well as instructional activities and end goals. A few of the teachers commented on their own growth in their understanding, and their appreciation for the learning trajectories as tools for teaching. These comments suggest that some of the teachers were beginning to see themselves as learners of the children's learning trajectories. However, most teachers did not seem to be explicitly aware of their own movement on a trajectory of adult learning toward building a robust mental model of preschoolers' mathematical thinking.

Conclusion and Implications

The majority of teachers in this study, when asked how they changed after participating in LT-based PD and LT-based curriculum implementation, described changes in themselves relating to their new awareness of the mathematical capabilities of four-year-olds. Early childhood teachers who are knowledgeable about young children's probable natural mathematical development, mathematical end goals, and the mathematical tasks, activities and materials that facilitate advancement toward those end goals, may become equipped to bring out surprising mathematical capabilities in preschoolers. Even with this limited data set of written self-reports on teacher change, there was a noticeable dominance of unprompted teacher comments on their newfound awareness of young children's capabilities, despite their average of twelve years of teaching preschoolers. These teachers, supported by LTPD and LTBI, were learning from their students. Teachers' raised awareness of four-year-olds' capabilities may spur them toward more deliberate construction of their own mental models of young children's mathematical thinking.

Perhaps unknowingly, teachers who took part in LTPD had already embarked on this pathway. They had already learned some MKT (Ball et al., 2008) relevant to preschool mathematics; they had learned of mathematical domain goals and probable pathways of preschoolers' progressions toward the domain goals, as well as instructional tasks appropriate for preschoolers at varied levels (as evidenced in this study, and documented in observational studies, see Clements et al., 2011; Sarama & Clements, 2013; Sarama et al., 2016). The professional development was focused on helping teachers formatively assess young children and their mathematical abilities, and to modify activities on this basis to meet their needs. One of the teachers, in describing change, wrote: "I get better and adapt more every year". Most of these teachers, however, appeared unaware of their own developmental trajectories or journeys toward building a strong mental model of preschoolers' mathematical thinking. Perhaps teacher educators could do more to explicitly describe PD end goals and adults' progress points in reaching them. That is, teacher educators might prepare teacher learning trajectories that delineate progress toward understanding children's mathematical thinking, thus strengthening the links between teaching, child learning, and teacher learning, while building the coherence that predicts teacher growth (Garet, Porter, Desimone, Birman, & Yoon, 2001). Just as LTBI may provide a theory for teaching (Sztajn et al., 2012), this data-driven theoretical inquiry points toward LTPD as a possible theory for teacher learning, with the child as teacher at the centre and the classroom as the learning laboratory, nurturing life-long learning

Support for this possible theory of life-long teacher learning about the pathways of young children's mathematical thinking, with the child at the centre, can be derived from Franke et al. (2001):

> We are not proposing that analysing children's thinking is the only avenue for teachers' growth to become generative; however, this focus does have characteristics that support generative growth. Children's thinking is available to teachers in their own classrooms daily: there are regularities in the strategies that children describe and principled ideas about these regularities… (p. 686)

Several caveats are important. The data were limited to a single survey of in-service teachers, therefore it represents a single point in time. Further, these are self-report data, with the inherent weaknesses of self-reports. Also, we were not able to conduct theoretical sampling with participants (thus, as noted earlier in this paper, we engaged in theoretical sampling of the literature). We believe, however, that the data extend, and are coherent with, previous research on TRIAD, and therefore make a modest contribution to the literature.

Although this study involved only in-service teachers, we believe the findings have implications for preservice teachers as well. Consistent with the AMTE Standards for Mathematics Teacher Preparation (Association of Mathematics Teacher Educators, 2017), early childhood teachers should be familiar with learning trajectories and how they can support teachers in understanding their students, as well as teaching them. Preservice teachers should be conversant with learning trajectories in general, and know at least one or two well, including having analysed videos (e.g., learningtrajectories.org) and applying them to clinical and classroom teaching experiences.

In summary, experience with the LTPD and LTBI generated fundamental change in teachers, as evidenced by teachers' responses (100% reported change) in this study, and as evident by observation (Sarama et al., 2016). They were more aware and appreciative of young children's mathematical capabilities and of the pedagogical use of all three components of learning trajectories: mathematical goals and content, developmental progressions, and instructional tasks and strategies (cf. Jacobs & Empson, 2016). Although the LTPD and LTBI catalysed these changes, they appeared to be enacted most completely as teachers interacted with children around the LT-based mathematics. This supports the use of learning trajectories in professional development and also provides guidance as to how LTPD affects teachers. That is, teachers' voices as documented here provide specific guidance to future efforts. In addition, future research should investigate how to incorporate such experience into diverse PD contexts and whether developing and explicitly sharing a trajectory of teacher learning would enhance teachers' learning in professional development settings. Another challenge is to investigate how such experiences could be provided to in-service teachers. As a final note, along with previous studies of teachers, including both self-report and classroom observations (Clements et al., 2015; Sarama et al., 2016), this study suggests that LTPD can have long-range benefits of interventions such as TRIAD for teachers. Too often, only child outcomes are considered.

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