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Contributions of "Mathematics for Elementary Teachers" Courses to Teaching: Prospective Teachers' Views and Examples

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A Mathematics course for elementary school teachers (MFET) is required in North America in most teacher education programs. Our study investigates the perceptions of prospective elementary school teachers with respect to the contributions of such a course to their teaching. The results show that acquiring an understanding of concepts from the elementary school curriculum is the main contribution that they perceive. We conclude with two perspectives – a pessimistic one and an optimistic one – on this finding.

Introduction

In many teacher education programs in North America a Mathematics course (or a sequence of Mathematics courses) for elementary teachers is either a requirement of, or a prerequisite for, entry to a teacher education program. Despite repeated calls for a "thorough rethinking of mathematics courses for prospective teachers of all grade levels" (Conference Board of the Mathematical Sciences [CBMS], 2001, p.6), and an agreement about the need for integration of mathematics and pedagogy at the elementary level in order to develop profound knowledge of mathematics for teaching (Ball & Bass, 2000), in many programs there is still the traditional separation between the Mathematics-content courses and the Mathematics-methods courses for prospective elementary school teachers. This study investigates prospective teachers' views of the contributions, both actual and potential, of the Mathematics-content course, referred to as MFET – Mathematics for Elementary Teachers – to their teaching.

On Teachers' Knowledge

With the extensive emphasis on teacher education in recent mathematics education research, the primary foci have been on assessing the knowledge that teachers have and exploring what knowledge teachers should have (e.g., Hill, Sleep, Lewis, & Ball, 2007; Davis & Simmt, 2006). Acknowledging that teachers' knowledge is multi-faceted, different attempts were made to categorize the components of such knowledge. Shulman's (1986) classical categories refer to subject matter knowledge [SMK], pedagogical content knowledge [PCK] and curricular knowledge. An extended categorization of teachers' knowledge was introduced by Deborah Ball and colleagues (Hill, Ball, & Schilling, 2008). It was referred to as "mathematical knowledge for teaching" [MKT] and presented as an extension of Shulman's categorizations. The PCK refinement included Knowledge of Content and Students [KCS], Knowledge of Content and Teaching [KCT] and Knowledge of Curriculum. The SMK refinement contained the categories of Common Content Knowledge [CCK], Specialized Content Knowledge [SCK] and Knowledge at the Mathematical Horizon. As knowledge acquired in a Mathematics-content course is of interest in this study, we note that CCK was described as shared among individuals who use mathematics, while SCK was considered as the domain of teachers that allows them "to engage in particular teaching tasks" (Hill et al., 2008, p. 377).

In contrast to this categorization, Huillet (2009) argued that the distinction between "purely" mathematical knowledge and mathematical knowledge adapted for teaching is not appropriate. Supporting this view, Bednardz and Proulx (2009) presented "intertwined mathematical didactical and pedagogical intentions" (p. 11) in teachers' work. Similarly, Davis and Simmt (2006) argued against the traditional separation of content and pedagogy, and claimed that "mathematics-for-teaching" can be considered as a distinct branch of the discipline of mathematics. Using complexity science they offered several intertwining aspects of mathematics-for-teaching, that included mathematical objects, curriculum structures, classroom collectivity and subjective understanding.

Extending the research of Davis and Simmt (2006) on what teachers need to know, Askew (2008) examined research evidence for the mathematics discipline knowledge that primary teachers might need in order to teach effectively. He concluded that the exact nature of such mathematical knowledge is still not clear and suggested that teachers' mathematical sensibilities should be developed. As a framework of working with prospective teachers Askew recommended learning new aspects of the discipline including precision (rigor), generalization and romance. Teachers should be precise in their use of mathematical language in describing mathematical concepts and actions. They should be aware of the move toward the general, which goes beyond getting correct answers to recognizing and extrapolating patterns. Romance implies care and curiosity. Teachers have to care about mathematics, to recognize and acknowledge the role that mathematics plays in shaping the world. Askew suggested that the attention of mathematics educators should shift from what mathematics primary teachers should know to *why* they should know this mathematics. He also argued that a distinction between content knowledge and pedagogical content knowledge might no longer be helpful. However, as mentioned above, despite the repeated claims against separation of content and pedagogy in teacher education and research, this long-established separation still exists within the coursework towards teachers' certification.

Sanchez and Garcia (2008) pointed out that mathematical knowledge of prospective teachers is one of the major concerns in mathematics teacher education and that many prospective teachers lack sufficient mathematical background. These researchers stressed that the mathematical activities in which prospective elementary school teachers should be involved include defining, justifying, and modelling. They also recommended that mathematical content be organized by major subject areas: Analysis, Geometry, Algebra, Statistics and Probability. These content areas and the activities constitute two critical dimensions for the instructional design of courses for prospective elementary school teachers. We elaborate on the content and nature of mathematics courses for elementary school teachers in the next section.

Several recent studies have focused explicitly on the usage of mathematical knowledge in teaching (e.g., Adler & Ball, 2009). This is also the focus of our investigations. Our prior research investigated the perceptions of secondary school mathematics teachers on the use of their knowledge of 'advanced' mathematics – knowledge they acquired during their studies at colleges and universities – in their teaching practice (Zazkis & Leikin, 2010). The results varied significantly: some teachers claimed that they have never used what they learned in their university courses in their teaching, while others claimed that they used it "all the time". Furthermore, even teachers who believed that they used their 'advanced' knowledge extensively had difficulty in providing explicit examples of this usage in relation to the school curriculum. The current focus on elementary school teachers, the mathematics they study in their university courses and its perceived usefulness for teaching is a natural follow up.

Mathematics for Elementary Teachers Course

While certification at the secondary level requires teachers to acquire a significant background in the subject matter, usually a degree or at least a minor, for elementary school teachers, as generalists, the mathematical subject matter requirements are limited. If such a requirement exists, it is usually for one or several mathematics courses designed specifically for this population. That is, such courses cannot be taken for credit for a Mathematics or Science degree.

A typical Mathematics-content course for elementary teachers – and we infer what is 'typical' from a variety of textbooks with similar tables of contents (e.g., Bassarear, 2007; Beckmann, 2007; Bennett & Nelson, 2006; Billstein, Libeskind, & Lott, 2009; Musser, Burger, & Peterson, 2006; O'Daffer, Charles, Cooney, Dossey, & Schielack, 2007; Sowder, Sowder, & Nickerson, 2010) and a variety of course outlines or course syllabi posted on the web – provides an overview of the underlying concepts of elementary mathematics. Typical topics include number systems and algorithms, patterns and introductory number theory, measurement and geometry, probability and data analysis. Different authors and publishers, in an attempt to satisfy the market, chose a combination of different perspectives on concepts and topics, such as problem solving, mathematical reasoning, the use of manipulatives, connection to the *Standards*, or the use of technology. The degree to which a certain perspective is implemented depends on the instructor's choice; however, the core topics remain the same.

The Study

Participants

Participants in this study were prospective elementary school teachers (PTs) enrolled in a one-year teacher education program that follows the completion of a Bachelor's degree. All of the PTs had some teaching experience in elementary school, having completed a 'practicum' of either 6 weeks or 5 months. (In the institution where this study was carried out there is a requirement to complete a "short practicum" of 6 weeks and a long practicum of 3.5 months during the teacher education program.)

All of the PTs had taken a MFET course, similar to the 'typical' course described above, as it is a required prerequisite in their teacher education program. They completed their MFET course at various times, with various instructors and at various colleges or universities. As such, our study concerns the course, rather than its specific implementation.

Research Questions

Our study attempts to address the following question:

How do prospective teachers describe the contribution of their MFET course to their teaching? Or, stated differently: What have prospective teachers learnt in their MFET course that they perceive as useful for their teaching?

Data Collection

During the data collection all of the participants were enrolled in the beginning of a Mathematics-methods course (Note: A Mathematics-methods course is taken during the program, in contrast to a MFET – Mathematics-content course, which is a prerequisite to the program). The data included a written response task and clinical interviews.

A written response task was administered to a group of 25 PTs. Initially they were asked to provide examples of several teaching situations in which their mathematical knowledge from the MFET course could have been useful. The teaching situation could have been actual or imaginary. However, several students claimed that they "just knew things" and had difficulty identifying the source of their knowledge. Acknowledging this difficulty, the written response task was modified. The task sought examples of usage, either actual or potential, of mathematical knowledge beyond the topic that was being taught.

Individual interviews were conducted by a research assistant with 14 PTs, volunteers from a different group. The interviews initially attempted to solicit explicit examples of mathematical knowledge usage in teaching elementary school mathematics. That is, after a brief introductory conversation about when and where the interviewee was enrolled in a MFET course, the interviewer explicitly asked the PT to provide examples of situations, where what was learned in an MFET course was used, or could have been used, in teaching. However, the flexible structure of the interviews turned in part to a conversation about the MFET course in general and its contributions to teaching.

Data Analysis

The data analysis was ongoing, using a qualitative approach based on grounded theory procedures and techniques (Strauss & Corbin, 1990). In the written responses to the task that explicitly addressed mathematical knowledge from the MFET courses we identified the mathematical topic in each example and the setting in which the example was presented. These are summarized in Tables 1 and 2, respectively. We then identified several recurring themes in the written responses, such as ability to explain or "explain better," and awareness of different approaches. Then, in the analysis of the interviews we identified additional emerging themes, such as mathematical connections and extending horizons. The themes that were identified in participants' responses serve as an organizing structure for the subsequent results and analysis.

Results and Analysis

To foreshadow our main observation, we start with an illustrative comment from Linda:

> I likely learned about […] in grade school. It was not until my MFET course that I learned the <u>reasoning behind why this works</u> and fully understood […]. This was very helpful when teaching […] during my practicum.

Please note that we intentionally deleted the mathematical content that Linda mentioned, as similar comments were provided with respect to different topics and procedures. We return to the ideas of "understanding" and "reasoning behind why this works" in our subsequent analysis. In what follows we attend to the recurring themes in participants' responses as well as to their particular examples of knowledge usage.

Initial Hesitation

In the clinical interviews, among the initial 'warm-up' questions, the participants were asked about their MFET course. The questions were of a general nature, such as, where did you take the course, how long ago was it, what did you think of it? Two typical responses are presented below.

> Rita: Actually when I enrolled in it I was so <u>freaked out</u> the first couple of days because I'm so <u>terrified</u> of math that I was going to transfer out, but I really wanted my elementary school pre-req's and so I was in the middle of transferring out and then I just really <u>started enjoying it</u>, and I stuck with it, and I ended up getting an A, so I was like, this wasn't so bad.

> Betty: I really enjoyed it. I was <u>nervous</u> when I first went in because I hadn't done math since grade 11, so it really actually got me excited about math again. But that being said when I knew I had to do this I was a little bit <u>anxious</u>.

The words anxious, nervous, terrified, freaked out, intimidated, panicky and alike were common in participants' descriptions of their entry point to the course. However, many students reported a degree of confidence and satisfaction towards completing the course, as well as some joy and excitement. While a level of confidence with the mathematical content taught is essential for teaching, we were further interested in specific examples of what PTs believe they learned that was helpful for teaching.

Examples of usage

In this section we summarize examples of usage from participants' written responses. The particular examples that were provided in the interviews appear further on, as illustrations of the recurring themes. Out of 25 PTs who completed the written response task, 17 explicitly referred to the knowledge acquired in their MFET course, generating 42 examples of knowledge usage. Table 1 presents a distribution of the 42 examples by topic. Table 2 presents a distribution of the 42 examples by the intended usage.

Table 1
Distribution of examples of topic

Mathematical content	Number of examples
Computation (compatible numbers, shortcut tricks, estimation, order of operations)	9
Elementary Number Theory (divisibility, Fundamental Theorem of Arithmetic)	8
Fractions and Decimals	8
Geometry (area, perimeter, angles, pi)	8
Algebra	5
Other:	
Work with different bases	1
Pascal's triangle	1
Division by zero	1
TOTAL	42

Table 2
Distribution of examples by the intended usage

Method of usage	Number of examples
Evaluating correctness of a student's response	24
Helping a student, responding to a question	8
Creating examples/tasks/activities for students	5
Classroom management/grouping of students	5
TOTAL 42	

Computation. The most frequent category is that of computation, however the examples that we clustered there are very different. They include computational shortcuts and tricks, such as how to multiply a 2-digit number by 11, evaluating the potential correctness of a student's answer by estimation, and identifying the source of a student's error, such as disregard of the order of operations. Furthermore, the prospective teachers mentioned the possibility to consider compatible numbers by performing computations or estimation mentally, while their students learn computational algorithms. In most of these cases (24 out of 42) the knowledge from the MFET course was used in order to evaluate a response from a student by a method different from the one used by the student. Awareness and appreciation of different approaches is elaborated upon further in the analysis of the interviews.

Elementary Number Theory. In this category we clustered examples that referred to divisibility rules, prime numbers and prime decomposition. Familiarity with divisibility rules served teachers in recognizing an error in a student's answer, such as when the difference of two odd numbers was not even, or when the sum of two numbers divisible by 9 was not divisible by 9. It further served when designing examples for student work, for example, when students are learning the division algorithm, divisibility rules help the teacher to present students with exercises of division without remainder, without carrying out the division. Moreover, some PTs reported that divisibility rules helped them when planning an activity that involved student work in small groups, that is, deciding on whether equal size groups were possible with a given student attendance.

One example described a hypothetical situation in which students were surprised that a different starting point in building a factor tree led to the same final result. The teacher relied on her familiarity with the Fundamental Theorem of Arithmetic and suggested to these students to check yet another starting point. In fact, this was one of only two examples that referred to a teacher's knowledge from a MFET course that is usually not a part of elementary school mathematics.

Fractions and Decimals. Included in this category were examples related to introducing students to different models of representing fractions and equivalent fractions, to converting improper fractions to mixed numbers, comparing fractions and performing operations with fractions. Examples in this category referred mostly to designing pedagogy for student understanding and, again, evaluating student answers, achieved by applying a standard procedure, by different means, such as comparing fractions by attending to units and not to a common denominator.

Geometry. Examples related to geometry mentioned the sum of the angles in a triangle and in a quadrilateral, the concepts of area and perimeter, and the meaning of p. Knowledge from the MFET course was helpful in assessing students' work, for example, when students measured all the angles of a quadrilateral and calculated their sum to 274 degrees, the teacher immediately recognized an error either in measurement or in addition. Examples provided by PTs in this category also referred to designing activities to introduce concepts.

One example reported on an activity carried out with elementary school students that replicated an activity previously experienced in the MFET course. The activity involved measuring the diameters and circumferences of different circular objects in order to introduce p as a ratio and not "as a strange appearance in some formulas".

Algebra. In this category PTs' examples mentioned their knowledge of solving equations in order to obtain the answer "quickly" and evaluating the work of students that was performed without algebraic means.

Other. As "other" we refer to three examples that did not fit into any of the previous categories. One participant mentioned responding to a student question about division by zero, an explanation acquired in her MFET course. Another participant mentioned that his experience and personal difficulties when working with different bases in the MFET course made him a much more patient teacher and more understanding when dealing with student difficulties, and this was perceived as having major importance in his teaching. Another participant mentioned that identifying patterns in Pascal's triangle in the MFET course resulted in designing an activity for students that integrated mathematics and art. This was the second example of using knowledge from MFET that is not a traditional part of elementary school mathematics. (The example mentioned earlier related to the Fundamental Theorem of Arithmetic.)

The repeating themes identified in the written responses of PTs were those of understanding and awareness of different ways to perform a mathematical task. We elaborate on these themes in further detail, and add other themes, as we turn now to the analysis of clinical interviews.

Understanding and explaining

The most robust theme, which was mentioned in the written responses and which appeared in all of the interviews, was that of understanding. Only two participants mentioned that their MFET course helped with "revisiting" or "refreshing basic skills". The majority maintained that in their MFET course they understood mathematical ideas, in some cases for the first time, and this personal understanding was ultimately related to the ability to explain, that is, to teach. While Tanya (below) makes an explicit connection between understanding and teaching, Betty elaborates further, starting with a severe criticism of her elementary school experience and her desire to find out why certain rules exist.

> Tanya: I always struggled in math for myself and taking this course helped me understand math better. You definitely need to understand that to be able to teach that.
>
> Betty: She explained all of the things that we just were taught in elementary school as "this is the way it is", like this is the rule. She'd explain why. I remember when I was a kid, I was like why? Why is this the rule? And then they'd be like, because it is. And I'm like well, OK, that doesn't help. So she explained why those rules were and I was just like, yes, finally, I can understand

> it because it's true, if you understand why the rules exist your application of them will be more accurate but also you can figure out other rules. Do you know what I mean? You need to really understand why the rules exist *in order for you to teach* this.

Once the connection between understanding and teaching was mentioned, the interviewer invited specific examples of concepts or ideas that were "really understood" or "understood better" as a result of the MFET course experience. The mathematical topics mentioned in the interviews echoed those from the written response task. These included algorithms, fractions, focusing on division of fractions, multiplication tables, and measurement.

Understanding and explaining: *Algorithms for addition and subtraction*. Participants reported that their experience with bases-other-than-ten was eye opening. First, it reinforced their understanding of computational algorithms, and then it helped them gain appreciation of the difficulties that students may experience.

> Mark: Doing different bases. I remember thinking that that really helped break things down and to really get an idea of how to add and subtract and multiply…

> Lisa: It gave me an appreciation for how hard it can be to add when you aren't using base 10, and we're so used to it. I sort of take it like nothing is a given anymore. Children may not understand that after you get to 10 when you add up two columns of numbers that you have to take the 1, the groups of tens, and regroup to the next column over, so while we might think that that's intuitive, like when you get to 10 you carry or regroup, and it's completely not intuitive. It needs to be taught.

> Betty: That helps a lot though because I stopped thinking of numbers in terms of base 10, because it's just so automatic. What that really helps with is the carrying the one. Because you're not carrying one, you're carrying ten over. It's lots harder and it was really good because it made us remember what it must have been like to learn it the first time. […] I remember doing the carry the one, I don't remember ever, ever being told that I was carrying one group of ten over.

In these excerpts Mark makes a general statement that work with other bases helped in understanding the ideas behind common computational algorithms. Betty and Lisa are more explicit, describing a particular newly acquired understanding of "carry the 1" and its relationship to the groups of ten. The observation that work with other bases helps teachers understand, or understand better, the ideas and algorithms of decimal representation of numbers was a researcher's perspective suggested in prior research (e.g., Zazkis, 2008). However, it is notable to hear such a comment made by participants.

Understanding and explaining: Multiplication. The idea of multiplication was mentioned in three interviews.

> Maya: If you understand this, it's not that hard, the idea of multiplication – you explain how you start with addition and then a shortcut, not just something to stare at, you know, stare at the wall.

> Lisa: Multiplication tables – I thought that has to be a skill that's just memorization and drills and, actually, you can do a lot with understanding around that, like patterns and stuff.

Lisa and Maya both reflect on their school experiences with multiplication tables. For Lisa the learning of multiplication was by rote memorization. Maya mentions "staring at the wall", making a reference to the multiplication table that was displayed on the wall in her classroom. Having mentioned this experience, their new understanding of the concept also presents itself in a way of teaching. For Maya multiplication is a shortcut for repeated addition; for Lisa multiplication is connected to patterns.

Understanding and explaining: Fractions and Division of Fractions. Fractions are known to be among the more challenging topics in elementary school mathematics. While the "pie" is the most popular model in textbooks, several research studies have shown that different representations, such as a number line, may promote a better understanding of the underlying ideas (e.g. Lamon, 2001).

> Mark: I'm not stuck on pie any more.

This is Mark's illustrative description of his limited prior knowledge, based on the dominant part-whole model of fractions, as well as of acquiring different representations of fractions.

The idea of division of fractions, or division by a fraction, was mentioned by 11 out of 14 interviewees as one of those ideas that they "finally got" or "really understood" in their MFET course.

> Rita: I always remember you just invert and multiply by, so but I feel like we were kind of steered away from just memorizing the formula, so I don't want to tell my students oh, just invert and multiply. […] just understanding fractions like understanding that doing division of fractions gives you a bigger number […] like understanding the information I think will help me to teach it better because it's just like a foreign concept until you look at it and understand it. Unless you understand it I don't think you can teach it.

This excerpt exemplifies two important notions related to division of fractions: that the result of division is larger than the dividend, and that there is a reasonable justification behind the "invert and multiply" strategy. The former is a variation on a well-known misconception ("division makes smaller") held by students, which is the result of work with whole numbers and partitive view of division (Graeber, 1993). The struggle with the latter is frequently addressed in mathematics education research. The case of Ms. Daniels (Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992) has become a classical reference for a teacher with extensive mathematical background who fails to explain to her grade-6 students the standard division by fraction algorithm.

While resolving the "invert and multiply" puzzle was mentioned by most participants, Betty added another component to her understanding of division by a fraction.

> Betty: We were trying to come up with a simple word problem using the numbers 2? divided by ? and pretty much everybody came up with a word problem that was 2? divided by 4, not ? because everybody was writing word problems that used sharing or what do they call it, partitive understandings of division but it doesn't work when you are dividing by fractions, which is the quotitive method of division, it is the only one that works and that's not normally the way we think of division. I never before realized the relevance of having two different understandings of division.

As Betty notes in this excerpt, there is a special kind of situation that involves division by a fraction and realizing this leads to a newly acquired understanding of two different kinds of division, partitive and quotitive. The difficulty in identifying a problem situation that Betty refers to is well documented in the literature (Ball, 1990; Simon, 1993) and is seen as troublesome not only for prospective elementary school teachers, but also for individuals with a stronger mathematical background.

Understanding and explaining: Measurement concepts and formulas. Understanding the meaning of area and volume concepts was mentioned in five interviews. Furthermore, participants talked about understanding formulas, that is, understanding "where they come from". Participants referred to making sense of the "length ¥ width" knowledge and "knowing what area is beyond length ¥ width". A popular example related to the formula for calculating the area of a triangle.

> Stephanie: You need an understanding of why you can divide it by 2. If you were to take that and reverse, it would make a perfect square, and understanding why.

Though Stephanie expresses the view that understanding formulas is important, there is a potential misunderstanding in her reference to a "perfect square". The gesture that accompanied Stephanie's words "take that and reverse" implied rotating a copy of a triangle over the mid-point of one of its sides. However, such transformation creates a parallelogram – which also explains the division by 2 in the area formula for a triangle – rather than a square.

Multiple perspectives/ways

Closely related to the theme of understanding is the theme of acquiring different ways to approach a mathematical task. Lisa makes this connection explicit:

> Lisa: I have a deeper understanding for it and then I can see lots of different ways of doing it.

A similar statement is made by Maya, while Grace elaborated further.

> Maya: I think maybe just helping to see things from a different perspective.

> Grace: I think what I've learned is to view things, like more than one method of seeing, like solving a problem. I think that would really help me in September because I realize, from my own experiences that there's only one way to do

> math, especially in high school, there's only one way and if I don't understand how they've done it, then I won't get the answer and if I don't get the answer, then that's not math. So I think allowing myself to see that there's more than one method to doing math, that would be really evident, so I'd look for ways to observe the kids and assess them in different ways, that would be really helpful I think.

Grace explicitly connects her learning of "more than one method of seeing" to her future teaching – the ability to observe and assess her students without imposing one approach. When the interviewees mentioned different ways, perspectives, methods or approaches, the interviewer invited specific examples. Several of these examples are presented below.

> Maya: Like how to do it, like different ways of adding the same kind of thing we were doing like getting to a base number or a reference number of whatever. So if like 13 plus 8, you go 13 plus 7 could be like 20, because you take 3 from the 13 and add it to the 7 and that's ten, and then 1 more.

> Tanya: Instead of long division you can take out groups of the number that you're dividing by so if you're dividing 480 by 17 you can take out any groups of 17, so one 17, you can take it out and minus it off there, and put 1 down and so you don't have to follow the rote procedure, you can do it in many different ways and still arrive at the same answer. And that's one thing that you wouldn't have thought of before.

> Rita: When multiplying decimals, like doing 3.7 times 5.2, they teach you to count the numbers, but you could pay attention to the number before the decimal point like for example 3 times 5, so it should be roughly around 15, so they can use their own prior knowledge of multiplication and then just kind of pay attention to the decimal, but it should be roughly around that number.

Maya describes how addition can be performed mentally by relying on compatible numbers, Tanya describes division as repeated subtraction and Rita describes how placing the decimal point can be inferred by estimation skills, rather than by counting digits. The PTs do not use the terminology that we introduced in the previous sentence, but their examples explicitly point to these "different" strategies, strategies that they acquired in their post-secondary studies.

In addition to personal awareness of different strategies or ways to approach a problem, some participants acknowledged the importance of understanding "how other people think".

> Betty: I think that I've learned, and this is sort of intimidates me, is that I have to open my mind to understand how other people think […] to not underestimate the students and to try really hard to understand how they came up with, to say show me how you did that, what is this, the trick is going to be able to do that just from their work and see what it is they're thinking.

And the connection here is obvious: personal awareness of multiple strategies is helpful in trying to understand students' strategies and approaches that may be different from the conventional ones (Leikin & Levav-Waynberg, 2009).

Questions and connections

The ability to acknowledge different approaches in students' work was also connected to the ability to deal with students' questions. Betty explicitly values her mathematics instructor's ability to deal with students' questions and she connects this to the teacher's knowledge, "you need to know your stuff".

> Betty: And so what I got from that, and that was quite awhile ago that I took that, and what stood out for me is that she was nice, well no, she wasn't nice, but she answered our questions very explicitly, she was able to understand what it was we were asking. You need to know your stuff to be able to do that.

Tanya relates her prior knowledge, acquired in part in her MFET course, to the ability to address the needs of students who are progressing faster than the rest. For Anne her knowledge serves in making connections, and for Betty it is the ability to explain why.

> Tanya: I also think it helps when they have questions, you'll have other knowledge to draw from for questions they may have. Some might be progressing further and want to know. They don't want to stop with what that activity is talking about, they want to know more. You need to have that background knowledge to answer those and to help them explore.

> Anne: What I have that they don't have yet are those connections, like they wouldn't necessarily think of it, why is it the same, why is the division symbol the same as a fraction?

> Betty: I don't want to be the teacher who says that's the way it is. [...] And I remember she told us why that works, I remember she did bring that up and I was like, oh, well that totally makes sense when you say it like that and I don't think that's above kids' understanding. I don't think that you need to force it on them but if they ask I'd like to be able to provide an answer, at least guide them in the direction of finding the answer, rather than just saying this is the way.

Note that Betty makes explicit reference to her mathematics instructor, "she told us why that works", whereas she herself balances the ability "to provide an answer" with her pedagogical belief of wanting to "guide them in the direction of finding the answer".

Extending horizons

Several PTs mentioned that the course opened their minds, extended their horizons and influenced their views on what mathematics is about.

> Lisa: I guess working in different number base systems, that <u>gives you a meaning</u> of numbers. So like with the ancient mathematics, like the Egyptian and Babylonian and I forget which ones, but we did like a base sixty system, and then we did binary for a bit. I hadn't ever heard of that before.
> Rita: We did base 6 and this helped a lot. This class <u>just opened up my mind</u> to numbers because you just think numbers are numbers, they're just there but you don't realize that we actually have created a system, like a number system,

> like a base 10 system, because I don't know, I just didn't understand the base 10 system before I did another system.
>
> Betty: Math isn't about numbers and plugging in numbers, it's about thinking about the solution. For me it's not $x+y=z$, it's thinking about how to get $x+y$ to $= z$.

Lisa refers explicitly to different ancient numeration systems, while Rita mentions her experience with numbers represented in different bases. In both cases this "gave meaning" or "opened up their minds" to a richer understanding of numbers. Betty expresses her view of not only what mathematics is (thinking), but also of what it is not (plugging in numbers). This illustrates her view contrasted either against her prior beliefs or against what she identifies as popular perceptions.

"Math today is different"

The idea that "Math today is different", that is, different from the mathematics PTs experienced as learners, was the second most prominent and unsolicited theme in the interviews, mentioned by nine participants. In the chosen excerpts this difference is contrasted with a description of prior experience, which is common among the participants.

Anne: Math is completely different today. I asked my teacher, I don't understand this and he's like well can you do it, that's how I remember most of my math being, all I learned was equations and if you could do the equation, if you could use the equation then you didn't need to know anything else.

> Cara: When I was going through math, it was just the numbers, not the problems.
>
> Maya: In my schooling, that's kind of what it was like. It was written up on the board and like, yep, OK, and then if you didn't understand they just did exactly the same thing over and over again. They didn't try to explain it, if you didn't understand it she just repeated exactly the same thing to you but a little bit louder.
>
> Betty: I very distinctly remember asking him a problem and he wrote the problem out on the board and showed me the answer, I was like I don't get it, so he did it again and he did it slowly and it's like, stop right here, how are you getting from here to here, it's like I don't understand this part and he's like well you just do this and it's like – no, how, how? And he's like, he just wrote the number down again and I was like, writing the number down six times isn't going to make me understand how you get that number, how do you get that number, what are you doing, like he didn't think about it anymore, you know what I mean, he just said this is what happens, and he got really frustrated and I was getting frustrated and I said, I was like, this isn't helping me learn, you aren't teaching me anything here and you're just showing me the answer and I don't need the answer, I need to know how to do it, and he got all mad and I got sent out of class.

At first we considered the theme of mathematics being different, and the presented prior experiences of frustration, as not explicitly related to our research questions. However, on a second look, acknowledging and appreciating this difference – between mathematics of yesterday and that of today or tomorrow – can be considered as the main impact of a teacher education program in general and a MFET course in particular.

Participants implicitly or explicitly contrasted their experience of "doing" and being shown how to do, with the desired explanation and understanding. The connection between understanding and teaching was explicitly acknowledged.

Discussion

Acquiring understanding is a declared purpose of the MFET courses. For example, Musser, Burger and Peterson (2006), authors of one of the popular textbooks for such a course, state explicitly in their introduction:

> This book encourages prospective teachers to gain an understanding of the underlying concepts of elementary mathematics while maintaining an appropriate level of mathematical precision. (p. xi, our emphasis)

In the "Message to prospective and practicing teachers" on the first pages of their book, Sowder, Sowder and Nickerson (2010) mention different perspectives and contributions to teaching:

> Some mathematics may be familiar to you, but you will explore it from new perspectives. […] Though the course is about mathematics rather than about methods of teaching mathematics, you will learn a great deal that will be helpful to you when you start teaching. (p.xiv, our emphasis)

As such, our findings suggest that a MFET course, or at least the offerings of the course that our participants were enrolled in at different times and at different institutions, achieved the set goal, at least from the perspective of participants in this study. However, this personal perspective of participants needs to be investigated further. It cannot be concluded from the participants' testimonials that they have indeed acquired a desirable level of what researchers referred to as PUFM – profound understanding of fundamental mathematics (Ma, 1999) or KDU – key developmental understanding (Simon, 2006), that is deemed as a prerequisite for MKT – mathematical knowledge for teaching (Silverman & Thompson, 2008).

Taking a pessimistic view on our results, we note that the majority of participants entering a teacher education program for certification at the elementary level acknowledged that they did not sufficiently understand the concepts and procedures of elementary school mathematics. Despite the small number of participants, this finding is significant, because at the university where this study was carried out the admission to teacher education program is very competitive. Only about 30% of the applicants are offered admission. Candidates have to demonstrate breadth and depth of academic preparation, a relatively high GPA in their degrees, their prior experience, possibly voluntary, working with children and youths, and to provide a written analysis of a videotaped classroom situation. So a lack of basic understanding of elementary mathematics exists among well-educated and intelligent university graduates. Our specific contribution, however, is in basing this finding on participants' testimonials related to their understanding (or lack of it) within particular concepts and topics of elementary school mathematics, rather than on researchers' observations.

Taking an optimistic perspective, we note that following a MFET course, PTs reported that they "really understood it", where "it" referred to various concepts, such as place value or fractions, or algorithms, such as column addition or division by a fraction. The utterance "really understand" repeatedly appeared in the interviews as well as in the written work. However, what does it mean to "really understand" something?

Betty summarized this as "knowing the reasons behind all the things that you teach the kids". Moreover, according to our participants, several related and further elaborated answers can be offered. For some PTs this means to know *why* and not only to know *how*, for others it means being able to provide an explanation to a student, and, furthermore, to be equipped with several different explanations. These views are in accord with the shift from what mathematics teachers should know to why they should know this mathematics, suggested by Askew (2008). The teachers' (partial) answer to "why" is the implementation of knowledge in teaching.

We mentioned above that self-report of acquired understanding does not necessarily mean that an appropriate level of understanding was achieved. Nevertheless, the personal acknowledgement of the importance of understanding and the ability to explain mathematics to students, rather than provide rules, is a valuable contributor to a teacher's comfort zone (Borba & Zulatto, 2010). Furthermore, the awareness that "math today is different", that is, that the desirable way of learning mathematics is different from the personal experience of participants, is an important step towards "teaching differently" or "teaching for understanding". We also suggest that the repeated reference to "different mathematics" may signify a change in personal beliefs of what mathematics is about. That is to say, it is not the mathematics that has changed, but rather the PTs' view of mathematics.

It is of interest to note that the themes of understanding and that "math today is different" – the two most frequent themes identified in the responses of PTs – are the only two themes that did not feature in the responses of secondary school teachers who described the usage of knowledge acquired in their post-secondary studies in their teaching (Zazkis & Leikin, 2010). This is hardly surprising, as it is likely that secondary school teachers take their mathematical understanding for granted and that they do not see "today's mathematics" significantly different from the mathematics they experienced as students. However, the themes of multiple ways or representations, of ability to address students' questions and make connections, which were identified in the responses of secondary school teachers in our prior research, echoed in the responses of PTs in this study. Furthermore, in both cases "extending horizons" was mentioned as a contribution of mathematics learned at the university level. However, while secondary school teachers referred to extending the horizons of their students, PTs in the current study referred to the contributions of their MFET course in expanding their personal viewpoints. This variation can be attributed not only to the difference between practicing (secondary) and prospective (elementary) teachers, but also to the self-perception of secondary school teachers as possessors of essential and broad knowledge.

Reflecting on the results of the current study, we also note that the MFET course is only one step in the mathematics education of prospective teachers, and it is likely that ideas that developed in this course are reinforced and re-examined in courses that attend to "methods" of instruction, that is, to pedagogy and curriculum. What can this imply for the teaching of mathematics? Anne's opinion is embedded in her question:

> Anne: And there were all those things that you learn that you're like, why didn't we just learn it like that from the beginning, because it would have helped me so much more.

In our optimistic perspective we would like to conclude with the hope that Anne's students will "learn it like that from the beginning".

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