Examining Preservice Secondary Mathematics Teachers' Responses to Student Work to Solve Linear Equations

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This study examined proposed teacher responses to students' work to investigate how they respond, what characteristics of a good response are more difficult than others to achieve, and whether particular student error types are more difficult to respond to appropriately. Sixteen preservice secondary mathematics teachers' proposed responses to five students' work to solve linear equations were analysed based on four characteristics of a good response: work toward student learning objective, draw on presented student thinking, draw on research on students' mathematical development, and leave space for students' future thinking. The preservice teachers' responses consistently met the last characteristic, but their skill at meeting the other characteristics differed markedly based on the type of student error in the work sample. An implication is the need to help preservice teachers learn how to address conceptual issues in their responses rather than solely focusing on procedural errors that are often irrelevant to meeting the learning objective.

Keywords · professional noticing · teacher preparation · responding to student work · linear equations

Introduction

Recent emphasis on eliciting student thinking and using it as the basis for instruction is predicated on a vision of mathematics teaching and learning that puts mathematical reasoning and sense-making at the forefront (Hiebert, 1997; Franke, Kazemi, & Battey, 2007; National Council of Teachers of Mathematics (NCTM), 2014). Researchers agree that while there is no single best way to respond to student thinking, the response the teacher gives should "help students deepen conceptual understanding while moving them forward toward procedural fluency and advanced mathematical reasoning" (NCTM, 2014, p. 54). Instruction based upon student thinking is significantly different than traditional lecture-based instruction focused on procedural skills that has been predominant in secondary schools (Hiebert et al., 2003; National Mathematics Advisory Panel, 2008). In traditional instruction, teachers typically respond to student thinking by praising correct answers or remediating mistakes by re-explaining the procedure, neither of which are aligned with a sense-making agenda (Crespo, 2002). These concerns have led many researchers to advocate that responding to student thinking is a specialised pedagogical skill which should be explicitly taught (Anthony, Hunter & Hunter, 2015; McDonald, Kazemi, & Kavanagh, 2013; Sherin, Jacobs, & Philipp, 2011).

In this paper we take up these issues and address the following research questions:

1. How do preservice teachers (PSTs) respond to students' work on solving linear equations?
2. Are some characteristics of a good response (e.g., working toward a student learning objective or leaving space for student thinking) more difficult than others for PSTs to achieve?
3. Are particular types of student errors more difficult than others for PSTs to interpret and respond to appropriately?

As emphasised by Schoenfeld (2011), professional noticing is a decision-making process which is a function of teachers' orientations, knowledge, and goals. Uncovering the nuances of teachers' professional noticing can help researchers better understand the complexities of responding to student thinking. There is little research detailing what makes this practice difficult for PSTs. Identifying strengths and weaknesses in PSTs' responses to student work as well as characteristics of student errors that are more difficult for PSTs to respond to appropriately can provide insights into the types of experiences that could advance teachers' responding abilities (Schoenfeld, 2011).

Literature review

Before detailing our study, we situate our work within the literature regarding responding to student thinking and the research on students' algebraic thinking. In the first section, we pay particular attention to research documenting how teachers respond to errors in students' written work. In the following section we describe research regarding difficulties students have in solving linear equations and connecting symbolic and graphical representations of functions, central ideas underlying the student work samples used in this study.

Responding to Student Thinking

Our work is situated within the growing literature base concerning professional noticing demonstrating that professional noticing of student thinking is a useful construct to both assess and develop skills needed for reform-oriented teaching (Anthony et al., 2015; Choy, 2016; Kazemi, Ghousseini, Cunard, & Turrou, 2015; Jacobs, Lamb, & Philipp, 2010). Jacobs and colleagues (2010) define professional noticing of student thinking as three interrelated skills of attending, interpreting, and deciding how to respond. Teachers must attend to mathematical ideas evidenced in student thinking and interpret that thinking in order to respond in a way that is consistent with the student thinking process and the relevant mathematics.

Teachers often struggle to respond in a manner that is faithful to both the mathematics and to students as mathematical learners (Ball, 1993; Lampert, 2001). In particular, teachers have a difficult time responding to student errors in a manner that provides space for further student thinking (Crespo, 2002; Son & Crespo, 2009), productively builds on students' current thinking (Jacobs et al., 2010; Son & Sinclair, 2010), and maintains a focus on key mathematical concepts (McDonald et al., 2013; Sleep, 2012), all of which are key components of a high-quality response (Jacobs, Lamb, Philipp, & Schappelle, 2011).

Leaving room for student thinking when responding to errors. Teacher responses to students' written work mirror the evaluation phase in the familiar Initiate-Respond-Evaluate (IRE) pattern (Mehan, 1985) that dominates classroom dialogue (Franke et al., 2007). Teachers predominantly respond to correct work with praise and to errors by demonstrating procedures or providing answers (Crespo, 2002; Milewski & Strickland, 2016). Such responses leave little space for independent thinking on the part of the student.

For example, Crespo (2002) documented how PSTs involved in a letter writing exchange with fourth grade students responded almost exclusively with praise or corrections. Although PSTs were more likely to include feedback on the students' thinking processes and ask follow-up questions toward the end of the eleven-week experience, praise remained the dominant response. When responding to students' incorrect written work, PSTs struggled to support students toward successful completion of problems without giving answers or entirely redirecting their thinking.

Son and Crespo (2009) examined PSTs' reasoning and responses to a student's non-standard method of dividing fractions. While PSTs' levels of reasoning were mixed, their responses to the student were primarily computational versus conceptual in nature and the predominant means of delivery was teacher telling. Interestingly, those PSTs who reasoned at a deeper level (e.g., were able to explain why the non-standard approach worked in general and how it was connected to the standard invert and multiply procedure) all resorted to telling. In contrast, PSTs categorised at the lowest level of reasoning, many of whom did not recognise that the alternative method was correct, gave student-focused responses that encouraged students to explain or justify their work.

Building on students' current thinking. Researchers investigating how elementary and secondary PSTs respond to student errors on proportional reasoning and geometry tasks found similar evidence of teacher-directed responses (Son, 2013; Son & Sinclair, 2010). The majority of PSTs resorted to "show and tell" responses (i.e., responses where the student is shown and told how to do it correctly) that focused on procedural aspects, even when PSTs correctly identified student errors as conceptually based. Based on patterns in PSTs' responses, Son & Sinclair (2010) identified three categories of communication barriers. The most prominent was over-generalisation, in which PSTs provided a general intervention that "ran ahead" of the student error. Alternatively, some PSTs responded in the other direction as if the student needed to be instructed on the basic underlying skills. Finally, many PSTs assumed the student knew the correct method(s) but simply forgot. In this case, PSTs dismissed the error by suggesting they would simply need to remind the student of the procedure or definition.

Linking student thinking to important mathematics. Another challenge teachers face when responding is maintaining a focus on the mathematics (Ball, 1993; Lampert, 2001; Sleep, 2012; Stein, Engle, Smith, & Hughes, 2008). Facilitating conversations that work toward a particular mathematical goal are especially difficult for PSTs (Anthony et al., 2015; McDonald et al., 2013; Sleep, 2012). In a study of secondary PSTs, Sleep (2012) documented several difficulties associated with steering instruction toward the mathematical point. For example, a lack of understanding of the mathematical connections led PSTs to respond with general questions which led to mathematical tangents, or questions that were too narrow which resulted in a funneling discourse pattern (Herbal-Eisenmann & Breyfogle, 2005). This funneling interaction not only left little room for student thinking, but because of the procedural focus, left few opportunities for students to grapple with the underlying concepts. Other issues included using inappropriate language for the learner and providing incomplete or confusing explanations, both of which occur when PSTs make assumptions that certain mathematical ideas should be obvious to the student.

While content knowledge plays a foundational role in teachers' abilities to attend to and interpret the mathematics in student thinking and thus can be considered a prerequisite for focusing responses on key mathematical ideas, content knowledge alone is not sufficient (Bartell, Webel, Bowen, & Dyson, 2013; Son & Crespo, 2009; Son, 2013). Even when PSTs demonstrate deep conceptual understandings and can accurately identify conceptual errors evidenced in student work, they struggle to respond in ways that directly address mathematics concepts (Son

& Sinclair, 2010; Son, 2013). However, responding can be improved through facilitated interactions and collaborative discussions around student thinking (Kazemi & Franke, 2004; Tyminski, Land, Drake, Zambak, & Simpson, 2014). For example, Fernandez, Llinares, and Valls (2012) documented how an on-line discussion of student work on a proportional reasoning task significantly increased PSTs' level of noticing. The collaborative on-line format required that PSTs delineate relevant aspects of students' strategies and validate possible interpretations with others. These discussions led PSTs to develop more sophisticated responses that honed in on the underlying mathematical concepts.

In summary, research suggests that teachers typically respond by evaluating the correctness of the student's response then telling the student what to do. They rarely address conceptual aspects of student thinking. Neither content knowledge nor the ability to attend to and interpret student thinking at higher levels guarantees that PSTs are able to actively take-up student thinking or respond to errors in a manner that advances mathematical understanding. This literature review highlights the difficulty of crafting high quality responses and the need for more detailed research into what aspects of a good response are easier or more difficult to learn as well as characteristics of student work that influence teachers' abilities to respond well.

Mathematical underpinnings of solving linear equations

The student work the PSTs responded to in this study documented students' efforts (written and verbal) to solve three linear equations. We identified two key mathematical understandings underlying the solving of these linear equations: (a) understanding what it means to "solve" an equation, including what that solution represents; and (b) understanding connections between graphical and symbolic representations of functions.

Understanding what it means to solve an equation. Understanding the equal sign from a relational versus computational perspective is a critical factor in correctly solving linear equations (Booth, Barbieri, Eyer, & Paré-Blagoev, 2014; Knuth, Alibali, Hattikudur, McNeil, & Stephens, 2008; Welder, 2012). Research has highlighted the importance of moving students beyond seeing the equal sign as a symbol to compute toward understanding that the equal sign indicates the equivalence between two expressions (Blanton et al., 2015; Carpenter, Franke, & Levi, 2003). Successful completion of the tasks used in this study (i.e., solving linear equations with one, none, and infinitely many solutions) can be supported by a shift in the conception of an equation from a statement about unknown numbers, to a question about the comparison of two functions over the domain of the real numbers. As Huntley, Marcus, Kahan, and Miller's (2007) study demonstrated, students overwhelming rely on symbolic manipulation to solve linear equations. While symbolic manipulation was generally effective for correctly solving equations with a unique solution, the majority of students lacked the conceptual understanding or adaptability necessary to interpret symbolic work that lead to identities or contradictions such as $0 = 0$ or $13 = 0$. Indeed, only 26 of 44 pairs of high school students working cooperatively to complete the given tasks were able to independently solve an equation with no solution and only 19 pairs reached a correct answer for the case with infinitely many solutions. Moreover, less than one-third of students approached the problems graphically prior to probing, which leads to our next big mathematical underpinning.

Understanding connections between symbolic and graphical representations. Students' limited understanding of the connections between equations and graphs has been well documented (Huntley et al., 2007; Knuth, 2000). Despite the prevalence of graphing calculators, students do not understand that the equation and its graph are two representations of the same underlying function and fail to recognise a point on the graph as a solution to its corresponding equation (Van Dyke & White, 2004).

Knuth's (2000) study revealed high school students' superficial understanding of the relationship between a graph and its corresponding equation. When presented with problems in which the graphical representation could be used most efficiently (e.g., asked to name a solution to the equation when the corresponding graph was provided) students predominantly relied on the symbolic representation and considered the graph superfluous. Even when prompted to use an alternative method, less than one-third of students considered using the graphical representation. This lack of adaptability was replicated in Huntley and colleague's (2007) study in which students had difficulty connecting their algebraic solutions to graphical representations of the solution, especially in cases with no solution or infinitely many solutions.

Methods

Participants

Sixteen secondary mathematics PSTs participated in the study. All were enrolled in their requisite secondary mathematics methods course at two U.S. universities during the Fall 2015 semester. The participants were finishing their teacher preparation program, with nearly all participants student teaching the following semester. They had each completed approximately 60 hours of practicum experience in secondary classrooms.

Context

The responding work highlighted in this paper was part of a larger sequence of activities designed to develop PSTs' professional noticing skills. Initial activities focused on the skills of attending and interpreting student work in the context of conducting a clinical interview with a secondary student. A thorough description of these activities can be found in Lesseig, Casey, Monson, Krupa, and Huey (2016) and Krupa, Huey, Lesseig, Casey, and Monson (2017). Subsequently, we designed instructional activities to develop PSTs' abilities to respond to student work. Instruction addressed crafting responses that met the four characteristics of a good response (Figure 1).

Characteristic	Description
1	Works toward student learning objective
2	Draws on and is consistent with the student thinking presented
3	Draws on and is consistent with research on students' mathematical development
4	Proposed interaction with student leaves space for student's future thinking (not just teacher's thinking)

Figure 1. Four characteristics of a good response.

These four characteristics align with the documented difficulties teachers have when responding to student thinking and build on the work of Jacobs et al. (2011), who noted Characteristics 2-4 as desirable in teachers' responses to students. The first characteristic aligns with a modern view of teaching and learning where teaching involves intentionally designing activities so that students learn specific, planned objectives (Ball & Forzani, 2009; Hiebert, Morris, Berk & Jansen, 2007). We subscribe to this view of teaching and learning; hence we added the first characteristic to emphasise the need for the teacher's response to progress the student towards meeting the learning objective. Additionally, this was a characteristic previously lacking in our PSTs' pre-instruction responses.

Data

This study's data consisted of PSTs' responses to five student work samples on problems from the Solving Equations interview protocol (Lesseig et al., 2016). Each PST previously used the protocol to conduct a task-based interview with a student, so the PSTs were intimately familiar with the protocol and had ideas about how students respond to the problems. The student learning objectives for the problems, relevant for Characteristic 1 of a good response, were **Objective 1**: Students will be able to solve systems of equations through multiple representations; **Objective 2:** Students will be able to explain in multiple ways (verbally, symbolically, and graphically) the significance of having two equations set equal to one other and the meaning of the solutions (students may also use a table to explain their thinking). These objectives were provided to the PSTs with the student work samples.

Figure 2 presents the three problems used in the protocol (along with their solutions), the accompanying five student work samples that were presented to the PSTs, and an interview transcript of each student solving the problem. All names are pseudonyms.

These samples were purposefully chosen to encompass student work on all three problems in the protocol and highlight a variety of reasoning strategies students may use when solving linear equations. All of the samples show students using symbolic manipulation to work the problems, as that was the initial approach of all the students and is the dominant strategy used by students (Huntley et al., 2007); however, none of the students used this method to correctly solve the problems. Thus, one goal of a teacher's response is to move each student toward successfully solving the equation symbolically (addressing Characteristic 1).

Problem	Student	Written Student work	Transcript
A: Solve $2x+3 = 5x-9$ [Solution: $x = 4$]	Zander	A. $2x+3=5x-9$ $-3 \quad -3$ $\frac{2x}{2} = \frac{5x-12}{2}$ $x = \frac{5}{2}x - 6$	Interviewer: What are you thinking as you're doing that part? Zander: To get x by itself, but then there is another x over here. Interviewer: So is x by itself? Zander: This one is [points to left side of equation], but not this one [points to right side of equation]. Interviewer: Why not? Zander: Because it's $(5/2)x$ – 6 and I don't know how to get rid of them.
B: Solve $2(3x+4)=6x+8$ [Solution: All Real Numbers]	Mark	B. $\frac{\cancel{2}(3x+4)}{\cancel{2}} = \frac{6x+8}{2}$ $3x+4 = 3x+4$ $-3x \quad -4$ $0x = 0$ $\boxed{x=0}$	Mark: Yeah, okay. So, for, so I want to get rid of this 2. That's like on the side, so I'll divide this whole other side by 2. I'm dividing the other side by 2 also to get rid of the 2. So then I get $3x$ plus 4 equals $3x$ plus 4. Because you divide 6 by 2 and 8 by 2. So then I can just subtract 4 from 4 which is 0 and subtract $3x$ from $3x$ which is zero. So… x = 0? Right? Cause you just divide 0 from 0, so x = 0?
	Trisha	B. $2(3x+4)=6x+8$ $6x+8 = 6x+8$ $6x = 6x$ $x = 1$	Trisha: Okay. Alright, so, I'm going to begin by… um… multiplying this parenthetical equation here, so it becomes…8…interesting. [laughs] Okay. Well, that's obvious… [Looks at her result, which is $6x = 6x$, for about 30 seconds.] Okay…Well, this one is confusing me, because they're equal on either side now, so… um… x is… 1? Or…?... I don't know. It's fooled me.

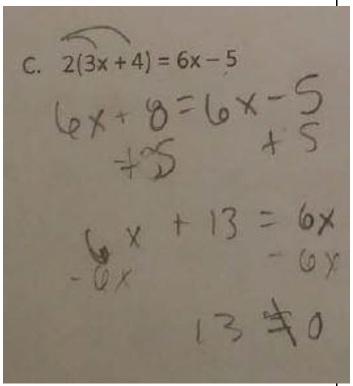
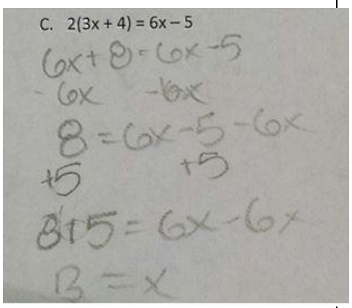
C: Solve 2(3x+4)= 6x-5 [Solution: No solution]	Laquisha		<u>Laquisha:</u> So, I wouldn't want to divide this by 2, cause 5 divided by 2 is kinda, kind of messy, so I think I'll just do the distributive property again. 6x +8 = 6x-5, yup. And then, minus 8. Well, actually- if it's a minus 8, I can just add 5, that's easier. 6x +13 = 6x. Umm, minus 6x, minus 6x, so it's just 13 = 0, so that's…Either I did it wrong, or that's just not equal to something. Or I could've…(concentrates)….Yeah. <u>Interviewer:</u> So, is this your solution? (points at '13 =0') <u>Laquisha:</u> Umm, kind of (Laquisha then draws a line through the equals sign to indicate 13 does not equal 0).
	Martha		<u>Martha:</u> Um, so I wanted to get the x's on one side and so I subtracted six x from both sides. Then I…then you had to get all of the constants on one side so I added five to both sides. Then I got eight plus five equals six x minus six x, and so I got thirteen equals x.

Figure 2. Student work samples.

Zander's work on Problem A showed that he likely knew how to symbolically solve a linear equation with a variable on one side but did not know how to adapt that procedure to a situation where the variable appears on both sides, a common sticking point for students (Kieran, 1992). Mark and Trisha's solutions of x = 0 and x = 1 to Problem B were selected because they are the most common incorrect answers to the problem (Huntley et al., 2007). Both students came to points in their solution process where they had identities (3x+4 = 3x+4 for Mark and 6x = 6x for

Trisha), but their reasoning diverged from there. Trisha studied the identity for thirty seconds, then said she was confused because "they're equal on either side now." She concluded that the answer was x=1, perhaps due to her belief that there must be a variable in the solution- a common belief of students (Sfard & Linchevski, 1994). Mark, on the other hand, did not recognise the equality of the two sides. He proceeded to work, getting an incorrect answer of $x = 0$ from the division of zero by zero.

Problem C is a linear equation that results in a contradiction, and thus has no solution. Both Martha and Laquisha's work show that they were similarly working symbolically to solve the equation without recognising the contradictions they were writing (e.g., $6x+8 = 6x-5$). At the end of the process, they diverged in their work. Martha decided that $6x - 6x$ was x, which produced the solution $x = 13$. Similar to Trisha, she may have decided that $6x - 6x = x$ in order to produce a solution of the expected form: x = [a number]. Her use of faulty symbolic manipulation is representative of what occurred for the majority of the students in the Huntley et al. (2007) study. Laquisha's symbolic work was correct, producing the statement 13 = 0, but she did not know to interpret it. Interpreting the statements that result from symbolic manipulation of linear equations with no solutions (e.g., 13 = 0) or an infinite number of solutions (e.g., 0 = 0) is challenging and most students do not come to the correct conclusion regarding the solution of the equation (Huntley et al., 2007; Sfard & Linchevski, 1994). Thus, Laquisha's work was selected because it represents a typical response when students reason about linear equations that have no solutions.

In a homework assignment, PSTs were directed to propose a response to each of the five students. These responses comprise the corpus of data for this study. One of the PSTs only wrote responses for Mark and Martha; thus the data consists of 16 responses for Mark and Martha and 15 for the other student work samples.

If the PSTs determined they needed to ask the student questions to understand what the student was thinking, they were to provide the questions with justification. Otherwise, the PSTs were to craft a response to the student that met the four characteristics of a good response (Figure 1). The PSTs had read the Huntley et al. (2007) article addressing research on students' mathematical development pertinent to solving linear equations, applicable for meeting Characteristic 3.

Analysis

PSTs' responses to the five student work samples were analysed based on the four characteristics of a good response (Figure 1). To perform the analysis, we described necessary features of a response to meet each characteristic for each of the five work samples. The descriptions for Characteristics 1-3 are presented in Figure 3. As all of the students were inclined to solve each problem symbolically but none did so successfully, understanding how to solve the problem using a symbolic method was the end goal for each of these students (Characteristic 1 in Figure 3). This could be accomplished by drawing upon multiple representations (Objectives 1 and 2), as noted for Characteristic 3 in Figure 3. Characteristic 2 in Figure 3 notes the critical point(s) in the student's work that needs to be addressed in the response.

	Characteristic 1: Works toward student learning objective	**Characteristic 2: Draws on and is consistent with the student thinking presented**	**Characteristic 3: Draws on and is consistent with research on students' mathematical development**
Zander	Moves student to symbolically solve linear equations with variables on both sides	Connects to his correct symbolic manipulation of integers	Connects to solving with alternative representation (graphical or tabular) or recognises that Zander thinks he can solve it just like $ax+b = c$
Mark	Works towards understanding symbolic solution method for solving linear equations that are identities	Addresses $0x = 0$	Connects to solving with alternative representation (graphical or tabular) or questions Mark regarding interpreting $0x = 0$
Trisha	Works towards understanding symbolic solution method for solving linear equations that are identities	Connects to $6x = 6x$ (and possibly the $x = 1$)	Connects to solving with alternative representation (graphical or tabular) or questions Trisha about conclusion of $x = 1$ (does she think she needs an answer of a number?)
Laquisha	Moves toward being able to symbolically solve a linear equation with no solution	Connects to $13 = 0$ and/or $6x+8 = 6x-5$	Connects to solving with alternative representation (graphical or tabular) or questions about the meaning of $13 = 0$ or reasoning about $6x+8 = 6x-5$
Martha	Moves toward being able to symbolically solve a linear equation with no solution	Addresses $6x-6x = x$ vs. $6x -6x = 0$ and interpreting $13 = 0$	Connects to solving with other representation (graphical or tabular) or questions Martha about conclusion of $x = 13$ (does she think she needs an answer of a number?)

Figure 3. Necessary features of the response to each student's work sample, by characteristic.

For Characteristic 4, leave space for student's future thinking, the agreed upon description "do not tell student how to correctly solve the problem or the answer" was used for all samples.

Using these descriptions, the data were coded independently by two researchers. Every response was given a score of 0, 1, or 2 on each of the four characteristics of a good response. Zero indicated no evidence, 1 designated partial evidence, and 2 noted complete evidence that the characteristic was addressed. To illustrate, Figure 4 presents excerpts of PST responses to Mark that received codes of 0, 1, and 2 for Characteristic 1 along with justification for those codes.

Participant Response to Mark	Coding for Characteristic 1: Works toward student learning objective
I would work with Mark to help him understand what it means to have the same expression on both sides of the equation. I would ask him to go into detail about what he means when he says to divide by 0 on both sides. Hopefully this would trigger for him that it is not possible to divide by zero. Maybe this would lead him to think about what $0 = 0$ means. If not, I would have him take a couple of steps back to when he had $3x = 3x$. I would ask him what happens when he subtracts the $3x$ from both sides. Maybe there is a different step that is more helpful? Ideally, this would trigger him to divide by three on both sides to get $x = x$. I think that once he is here, he might make the jump to all solutions (sometimes with a push, "what numbers for x make $x = x$ true?"). (Participant 3)	2: Mark is prompted to interpret his written statements from solving the equation symbolically to understand that the equation has infinitely many solutions. There is a consistent focus on the meaning of the identity statements.
I would ask Mark where the x's on the right side went because it could lead him to getting 0=0 rather than $0x$=0. This might help him realise that x doesn't equal 0. I would also ask him what he means by dividing by 0 in hopes that he remembers that you cannot divide by 0. (Participant 1)	1: Response proposes getting Mark to change $0x$=0 to $0 = 0$ in order to realise that his answer of $x = 0$ is incorrect, but does not support Mark in interpreting $0 = 0$.
First, I would ask "Can you think of another method of solving that would work?"… This may cause Mark to recognise his error in the first attempt, then I would know that it was a simple calculation mistake. If this doesn't provide any new information, I would ask "Does your answer seem reasonable? Why or why not?" to have Mark describe the meaning behind his answer, $x = 0$. (Participant 16)	0: Proposed response does not move Mark toward understanding solving this equation (or other identities) symbolically. If Mark solves the equation using another method, it is not proposed that this work be tied back to solving it symbolically. The question regarding the reasonableness of the answer is unlikely to move Mark forward to realise that zero is not the only solution to the equation.

Figure 4. Illustration of coding responses to Mark for Characteristic 1.

Following this initial coding, the researchers compared their coding scores and discussed any that differed until a consensus was reached. Finally, the coding scores were entered into a spreadsheet and basic numerical analyses (e.g., calculation of means, tallying in two-way tables) were completed.

Results

The results presented in this section answer the study's three research questions. We attend to the first research question, "How do PSTs respond to students' work on solving linear equations?" throughout the results section by providing descriptions and quotes of PSTs' responses. The second research question regarding whether some characteristics of a good response are more difficult than others for PSTs to achieve is addressed in the first results section, "Overall Results." The next section, "Analysis of variation in results by student work sample," answers the third research question about particular student errors that may be more difficult than others for PSTs to make sense of and respond to appropriately.

Overall Results

Table 1 presents the mean overall scores by characteristic (first row of table), as well as the mean scores on the characteristic coding for each of the student work samples.

Table 1.
Coding results: mean scores (overall and on each student work sample) by characteristics of a good response

	Characteristic 1: Works toward student learning objective	Characteristic 2: Draws on and is consistent with the student thinking presented	Characteristic 3: Draws on and is consistent with research on students' mathematical development	Characteristic 4: Proposed interaction with student leaves space for student's future thinking
Overall mean	1.35	1.34	1.49	1.94
Zander	1.27	0.93	1.53	1.87
Mark	0.69	1.06	0.94	1.88
Trisha	1.73	1.80	1.73	2.00
Laquisha	1.73	1.73	1.73	1.93
Martha	1.31	1.19	1.50	2.00

The overall mean score of 1.94 for Characteristic 4 was the highest of all the characteristics of a good response. Looking at the coding results for each student work sample shows that Characteristic 4 had the highest average score across all student work samples as well. Table 2 displays the breakdown by student work sample regarding the codes assigned to the participants' responses on Characteristic 4.

Table 2.
Coding results Characteristic 4: frequency (and percentage) of coding scores, by student work sample

	0-No Evidence	1-Partial Evidence	2-Complete Evidence
Zander	0 (0%)	2 (13%)	13 (87%)
Mark	0 (0%)	2 (13%)	14 (87%)
Trisha	0 (0%)	0 (0%)	15 (100%)
Laquisha	0 (0%)	1 (7%)	14 (93%)
Martha	0 (0%)	0 (0%)	16 (100%)

The prevalence of codes of 2 (complete evidence) and the high mean scores for Characteristic 4 demonstrate that across all the student work samples, the PSTs refrained from telling the student how to solve the problem or stating the correct answer. Instead, PSTs' responses allowed for the student to do his or her own thinking. The consistency in the scores across student work samples showed that the PSTs were able to reliably meet Characteristic 4 in their responses.

In contrast, the mean scores for Characteristics 1, 2, and 3 in Table 1 show that PSTs achieved a lower skill level on these characteristics. Moreover, the mean scores by student work samples showed there was much less consistency in their scores. To provide more information on the results for Characteristics 1 through 3, Tables 3 through 5 present the frequency (and percentage) of the coding scores by student work sample.

The results in Tables 3 through 5 make it clear that the PSTs' abilities to meet Characteristics 1 through 3 differed markedly by student work sample. Participants scored the highest on Laquisha and Trisha's responses and scored nearly the same across the three characteristics (both had mean scores of 1.73 on Characteristics 1 and 3; their mean scores were 1.73 and 1.8 on Characteristic 2). In general, the participants did quite well in crafting responses to Laquisha and Trisha that met the four characteristics of a good response. Scores for participants' responses to Zander and Martha were similar to one another and indicated that they had more difficulty meeting the characteristics of a good response. They particularly struggled with Characteristic 2, with mean scores of 0.93 (Zander) and 1.19 (Martha). Mark proved to be the most difficult student for the participants to respond well to. The scores on Characteristics 1 and 3 for Mark were markedly low, with means of 0.69 and 0.94 respectively, and were notably less than the scores for the responses to the other students on those characteristics. The scores regarding Characteristic 2 for responses to Mark were also low (average of 1.06), though the average for responses to Zander (0.93) was the lowest for that characteristic.

Table 3.
Coding results Characteristic 1: frequency (and percentage) of coding scores, by student work sample

	0-No Evidence	1-Partial Evidence	2-Complete Evidence
Zander	2 (13%)	7 (47%)	6 (40%)
Mark	7 (44%)	7 (44%)	2 (12%)
Trisha	0 (0%)	4 (27%)	11 (73%)
Laquisha	1 (7%)	2 (13%)	12 (80%)
Martha	2 (12%)	7 (44%)	7 (44%)

Table 4.
Coding results Characteristic 2: frequency (and percentage) of coding scores, by student work sample

	0-No Evidence	1-Partial Evidence	2-Complete Evidence
Zander	7 (47%)	2 (13%)	6 (40%)
Mark	6 (37%)	3 (19%)	7 (44%)
Trisha	1 (7%)	1 (7%)	13 (86%)
Laquisha	1 (7%)	2 (13%)	12 (80%)
Martha	5 (31%)	3 (19%)	8 (50%)

Table 5.
Coding results Characteristic 3: frequency (and percentage) of coding scores, by student work sample

	0-No Evidence	1-Partial Evidence	2-Complete Evidence
Zander	3 (20%)	1 (7%)	11 (73%)
Mark	4 (25%)	9 (56%)	3 (19%)
Trisha	1 (7%)	2 (13%)	12 (80%)
Laquisha	0 (0%)	4 (27%)	11 (73%)
Martha	1 (6%)	6 (38%)	9 (56%)

Further confirmation that the PSTs' abilities to craft a response that met Characteristics 1 through 3 differed by student work sample is found through comparison of the mean scores. The mean scores for Trisha (1.756) and Laquisha (1.733) were highest, showing the PSTs were most successful at crafting good responses to these student work samples. The mean scores for Martha (1.333) and Zander (1.244) were similar and approximately half a point lower than that of Trisha and Laquisha. This indicates the PSTs had less success meeting Characteristics 1 through 3 in their responses to Martha and Zander. Mark's mean score of 0.896 was substantially lower than the others, confirming that the PSTs had the most difficulty meeting Characteristics 1 through 3 in their responses to Mark.

Analysis of Variation in Results by Student Work Sample

Further analysis focused on understanding why PSTs' abilities to meet Characteristics 1 through 3 differed by student work sample. The participants consistently scored higher on their responses to Laquisha and Trisha. Looking for similarities in Laquisha and Trisha's work samples, we found that both students' symbolic work is correct (Laquisha writes the statement 13 = 0 and Trisha writes $6x = 6x$). The hurdle these students could not overcome was how to interpret these statements (e.g., what does a contradiction like $13 = 0$ imply about the solution to the original equation?). Therefore, a teacher's response to these students can focus on interpreting those statements, which the participants' responses generally did effectively. Often this was done by questioning the student about the meaning of the statement, such as Participant 6 suggesting to ask Laquisha "You got that 13 equals 0 and you know that this is not true. What could it mean or why do you think this happened? What does this say about x?"

Another commonality in the responses to these students was a suggestion to solve graphically: 11 of the 15 responses to Laquisha and 9 of the 15 responses for Trisha do so. Participants' awareness of the usefulness of multiple representations when solving linear equations was raised by their reading of Huntley et al. (2007) as well as the student learning objectives identified in the assignment. To receive a score of 2 for Characteristic 1, however, the response must include looping the student back to their symbolic work to interpret it after they have solved the equation graphically. This looping back is necessary for the student to learn to solve the equation symbolically, which is the method each student used initially to solve the equation, and to further develop students' understanding of the critical ideas of contradiction and identity in the context of solving linear equations. Approximately half of the responses that suggested to the student that she solve it graphically looped back to subsequently interpret her symbolic work (6 of 11 for Laquisha and 4 of 9 for Trisha).

The PSTs had more difficulty crafting high quality responses to the student work samples from Zander and Martha. Unlike the student work samples from Laquisha and Trisha, Zander and Martha take steps in their symbolic work which prevent them from solving the equations correctly. Responses to these students need to address those steps and move the student towards correctly solving the equation, which perhaps made the work of providing a complete response more challenging. For example, to receive a score of 2 on Characteristic 2 in a response to Martha, the response must address $6x$-$6x = x$ vs. $6x$ -$6x = 0$ as well as interpreting $13 = 0$. Table 4 shows that 50% of the responses addressed both of these aspects, while 19% addressed one of them and the remaining 31% addressed neither. Responses to Zander also need to attend to his symbolic work, as the verbal explanation accompanying his written work on Problem A showed that he likely knew how to symbolically solve a linear equation with a variable on one side but did not know how to adapt that procedure to this equation with a variable on both sides. However, five of the PSTs' responses did not identify that as the issue to address with Zander; instead, they thought he did not understand how to combine like terms or could not work with fractions (notably the 5/2 coefficient of x in the last line of his written work) and focused solely on addressing these points in their responses. Hence, all five of these participants received a score of zero for Characteristic 2. These details explain why Zander's mean score for Characteristic 2 was the lowest of all the samples. In addition, eight of the PSTs suggested that Zander make a graph to solve the equation, but none of them looped back to solving it symbolically. This contributed toward the lower scores in Characteristics 1 and 3 for responses to Zander.

Finally, we present an in-depth analysis of the responses to Mark, as the PSTs had the most difficulty responding to Mark in a way that aligned with the four characteristics of a good response. This analysis illuminates aspects of student work that can prove difficult for teachers to respond to appropriately.

Mark's symbolic work to solve the problem, though not complete in its documentation on paper (e.g., he does not write that $3x$ is subtracted from both sides of the equation), is correct through the step where he writes $0x = 0$. His verbal explanation shows that once he gets to this point, he pauses. All remaining statements are questions, implying Mark is unsure in his work. His final step of solving $0x = 0$ by dividing both sides by zero leads to an incorrect answer of $x = 0$.

The primary reason the responses to Mark scored lowest was that the PSTs never helped him understand how to interpret the statements in his work (e.g., $0x = 0$) to come to the conclusion that this equation is an identity and any number will solve it. The most common first thing addressed in the PSTs' responses to Mark was to deal with division by zero, done by Mark in his final step. This was addressed directly by six PSTs, who pointedly stated or asked if you can divide by zero, and indirectly by an additional two PSTs who asked why x is zero. Two participants chose to address division by zero as their second move, after first concentrating on $3x$-$3x = 0$ instead of $0x$. Thus, a total of ten PSTs addressed division by zero in their response to Mark. Addressing this is appropriate since it is mathematically incorrect. However, most of the PSTs stopped there, feeling their work with Mark was done once he knew that division by zero was not possible. For example, Participant 9 wrote "I would ask this student what happens when a number is divided by zero, because his algebra is correct for the entirety of the problem, he just commits this one critical error in the end. Hopefully this prompt would help the student catch his mistake." Only two of these ten PSTs continued on to assist Mark to correctly interpret statements in his work and conclude that the equation has an infinite number of solutions. Looking at other first responses to Mark, three PSTs focused on changing Mark's work of $3x$-$3x = 0x$ to $3x$-$3x = 0$. This was also attended to by two PSTs later in their responses to Mark. While this gets Mark to the statement $0 = 0$, none of these PSTs asked or supported Mark's interpretation of the statement. To illustrate, Participant 14's entire response to Mark was "I would like to ask Mark what it means to subtract 3x from 3x, and how or why that is different from subtracting 4 from 4." All of the aforementioned responses received low scores for Characteristic 1 because they did not complete the work that would be needed to help Mark understand the symbolic solution method for solving linear equations that are identities.

To meet Characteristics 2 and 3, a good response to Mark must address interpreting $0x = 0$ since this is the salient point where Mark has a correct statement but does not know to interpret it. Few of the PSTs noted this, with only four responses ever addressing what it means for two equations to be equal or more specifically addressing how to interpret $0x = 0$ or $0 = 0$ when solving an equation. Instead, the PSTs often thought the salient issue was to make Mark's written symbolic work complete, so they worked to get Mark to write that he subtracted $3x$ and 4 from both sides of the equation or as previously mentioned to change $0x$ to 0. Two of the PSTs asked Mark to rework the problem, distributing the coefficient of 2 on the left side first. Again, this response has the student working to symbolically solve the problem in the way the PST may prefer, but it does not address a mistake in Mark's work, nor does it address the relevant conceptual issue of interpreting identity statements in equations with an infinite number of solutions.

Regarding Characteristic 3, questioning Mark about the interpretation of $0x = 0$ is one way to meet this characteristic of a good response. The alternative route is to connect to solving the equation with other representations (graphical or tabular). This was done by half of the participants: one participant called for Mark to make a table, six participants asked Mark to make a graph, and Participant 11 asked Mark to make a graph and table. Yet, as seen with responses to the other students, few of them (2 of 8) connected the conclusions drawn from the graph or table back to the symbolic work.

Lastly, two of the PSTs' responses called for Mark to check his answer of $x = 0$ then asked him to see if other numbers for x would also work. While this assists Mark in seeing that there are other solutions and perhaps get him to the conclusion that there are an infinite number of solutions without telling him so outright, one would only know to try other numbers if he/she already knew the solution. Since this is not a viable path for solving linear equations in general, these responses were scored as 1 for Characteristic 1.

Discussion

This study highlights intricacies involved in responding to students' algebraic thinking. Specifically, we were interested in the relative ease or difficulty with which our PSTs could craft good responses to students' written work that met four characteristics: (1) works toward student learning objective; (2) draws on and is consistent with the student thinking presented; (3) draws on and is consistent with research on students' mathematical development; and (4) leaves space for student's future thinking. Our results indicate that PSTs were most successful at meeting Characteristic 4 in their responses and had more difficulty meeting Characteristics 1 through 3. Moreover, PSTs' abilities to meet these characteristics were dependent on the type of student errors involved.

The PSTs were consistently able to provide responses that left room for student thinking (Characteristic 4). The negative effects of teacher telling on student learning indicate how important it is for PSTs to be aware of responses that take over student thinking (Jacobs, Martin, Ambrose & Philipp, 2014). PSTs must be equipped with alternatives to telling that result in students making sense of correct mathematical procedures and ideas. Unlike Characteristics 1 through 3, PSTs' success in achieving Characteristic 4 was evident across all of the student work samples suggesting that their ability to refrain from taking over student thinking was independent of the type of student error presented.

In light of research demonstrating difficulties PSTs have in crafting responses that neither run ahead nor lag behind students' current thinking (Son & Sinclair, 2010; Sinclair, 2013), it is not surprising that Characteristic 2: draws on and is consistent with the student thinking presented, proved to be the most difficult. However, it is still promising that over 80% of PSTs achieved the highest ranking for Characteristic 2 when responding to Trisha and Laquisha and at least 40% of PSTs were able to do so for the remaining student work samples (those of Zander, Martha and Mark) (see Table 1 for details).

There is also considerable room for improvement in Characteristic 1: works toward student learning objectives. With the exception of responses to Trisha and Laquisha, less than one third of PSTs' responses completely addressed Characteristic 1. Recall that for this assignment PSTs were told that the learning objectives were for students to be able to solve systems of equations through multiple representations (**Objective 1**) and to understand the significance of having two equations set equal to one other as well as to explain the meaning of the solutions (**Objective 2**). Working toward these learning objectives proved most difficult when responding to Mark's work. More than half of the PSTs attempted to correct Mark's division by zero error but failed to move him toward an understanding of what the resulting statement 0=0 meant in terms of a solution set. These responses mirror the research demonstrating PSTs' tendencies to address procedural (instead of conceptual) issues when responding to student errors (Son, 2013; Son & Sinclair, 2010). This may be due to PSTs' acceptance of correct symbolic work as conceptual understanding (Bartell, et al., 2013), or lack of exposure to alternative responses (Son & Crespo, 2009). Given that previously most PSTs were able to identify limitations in students' conceptual understanding when interpreting student thinking (Lesseig et al., 2016), the latter explanation is more likely. However, either rationale indicates that PSTs would benefit from more illustrative examples of effective conceptual responses.

The PSTs' responses to Mark also highlight a general tendency toward providing a "quick fix" that may or may not advance students toward the intended mathematical goal or address underlying conceptual issues. PSTs often get distracted by mistakes and lose sight of the salient mathematics in a student's response (Crespo, 2002; Sleep, 2012). Overall, PSTs had more success maintaining a focus on key mathematical ideas (Characteristics 1 and 3) when responding to Trisha and Laquisha's work, perhaps because the algebraic manipulation was correct. Thus, it was less likely that PSTs would focus on procedures and could instead work toward helping these students understand conceptually what their symbolic solutions meant. PSTs more readily incorporated multiple representations into their responses to Trisha and Laquisha as well. They recognised that just talking about symbolic manipulation would not be sufficient, so they moved on to other representations. There is still room for improvement, however, as only half of the PSTs provided responses to Trisha and Laquisha that looped back to the symbolic work after introducing the graphical representations. Making connections across representations is an important step toward understanding the meaning of solutions achieved through symbolic manipulations (**Objective 2**).

Finally, it is worth noting that it was more difficult for PSTs to address conceptual issues when procedural errors were present. When students had trouble solving symbolically, PSTs fixated on correcting errors. Once students demonstrated proficiency with the symbolic work, PSTs deemed them ready to consider multiple representations. This tendency to think that students need to first demonstrate proficiency with procedures is problematic and runs counter to research demonstrating how conceptual understanding can support procedural fluency (Mathematics Learning Study Committee, 2001). Moreover, this traditional approach in which algebraic procedures are always presented first (Yerushalmy & Chazan, 2002) limits the mathematics that students are exposed to and presents a narrow, equation-based view of algebra. Instead we want to support PSTs' ability to integrate various representations and more importantly, to recognise when and how to provide conceptual instruction regarding procedures.

Implications

To move away from telling students what to do, PSTs need to not only be instructed on alternative ways of responding, but also be given multiple opportunities to practice and reflect on this skill. In-class work done prior to this assignment which encouraged using focusing questions (Herbal-Eisenmann & Breyfogle, 2005) with students and provided the PSTs with lists of probing questions they might ask supported the PSTs' success in meeting Characteristic 4 in their responses. However, even though the PSTs were able to respond in ways that left room for student thinking, the responses did not necessarily draw on the students' mathematical ideas or move them forward towards meeting the learning objectives. Moreover, PSTs' abilities to respond varied based on the type of errors present in the student work. These results point toward specific recommendations for teacher education.

PSTs need exposure to a range of conceptual and procedural errors students might present and opportunities to scrutinise this work with others. Ideally, mathematics teacher educators (MTEs) can facilitate conversations including weighing the affordances and constraints of possible responses to a student. Given the variation in PSTs' responses, MTEs also need to select tasks that provide accurate representations of PSTs' noticing abilities (Ding & Dominquez, 2016). Just as students' facility with a set of procedures doesn't guarantee mastery of a concept, PSTs' ability to respond appropriately to certain student errors does not guarantee they will always respond at an advanced level. Specifically, PSTs might need further support in deciding how to respond to student work that contains multiple errors or more subtle mathematical misconceptions. Assessing PSTs' noticing across a range of content areas and/or types of student errors can help MTEs design appropriate subsequent activities.

Content knowledge plays a critical role in what and how PSTs notice the mathematics in student thinking and are able to respond accordingly (Bartell et al., 2013; Ding & Dominquez, 2016; Son, 2013). Since we didn't directly assess PSTs' content knowledge, we make no claims concerning its impact on PSTs' ability to respond. However, both the interview protocol and the assigned reading from Huntley et al.'s (2007) research study allowed PSTs to reflect on and discuss key mathematical ideas underlying these linear equation tasks (e.g., what the solution to a linear equation represents, connections between symbolic and graphical representations of a function) as well as the difficulties students may have understanding these concepts. We contend that these opportunities for PSTs to collectively consider the mathematical entailments of the tasks can support the development of good responses. This background knowledge seems particularly critical in order to provide responses that are consistent with research on students' mathematical development (Characteristic 3) and work toward the mathematical goal (Characteristic 1).

While we recognise the value of exploring the conceptual underpinnings, learning progressions, and research on typical student errors relevant to the task students are completing, we also know that delving this deeply for every topic in the secondary mathematics curriculum is not feasible in a single methods course. However, it is our hope that by doing a deep dive in one content area, in our case reasoning around solving linear equations with no, one, and infinite solutions, PSTs will be better positioned to consider the salient mathematics when responding to student work in other content areas. In their work with PSTs, MTEs can draw attention to the fact that not all ideas are equally important mathematically (Leatham, Peterson, Stockero, & Van Zoest, 2015) and help PSTs identify conceptual milestones along a learning trajectory. Such work can further support PSTs' ability to distinguish among critical conceptual errors (e.g., ideas that if perpetuated might significantly misrepresent mathematics), easily remediated procedural mistakes, or tangential mathematical ideas and respond accordingly. Finally, to increase the likelihood that these responding practices are generalisable, we encourage MTEs to focus on research-based strategies for productive responses such as making connections across symbolic and graphical representations, emphasising meaning, or uncovering student thinking (Milewski & Strickland, 2016; Stockero & VanZoest, 2013).

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