Preservice and novice teachers' knowledge on preformal proofs: Triangle postulate as an example

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By considering the example of proving the triangle postulate, this study aimed to explore Hong Kong preservice and novice teachers' knowledge competencies and their beliefs about preformal and formal proofs. The findings revealed that such teachers are not proficient in using preformal proofs and do not realize that preformal proofs are a useful tool for connecting abstract geometrical concepts with concrete meanings. We conclude by providing strategies for teachers to use preformal proofs effectively in their teaching of geometric propositions.

Keywords · mathematical knowledge for teaching · preservice teachers · preformal proofs · teachers' knowledge for proof and proving · triangle postulate

Introduction

Among the various types of proofs in mathematics, proofs on geometric propositions are the most popular in secondary school syllabi. For example, the proposition: *the sum of the (interior) angles of a triangle is equal to 180°*, also known as the triangle postulate (Weisstein, 2003, p. 2147), acts as a foundation in Euclidean geometry and is fundamental to the Hong Kong secondary mathematics curriculum for geometry. Through the instruction of the triangle postulate, students are expected to attain a more sophisticated level of geometric thinking. Concerning this, Crowley (1987) commented the following:

> Too often, geometry is taught in a mechanical way. Consider the fact that the sum of the angles of a triangle is 180°. Frequently, this fact is established by generalizing after measuring the angles of a few triangles, or worse, students are simply told the information. (p. 13)

Crowley (1987) further indicated that students could understand a geometric property both inductively and deductively and that they could develop geometric proficiency by proving a geometric property through a nonstandard proof based on tessellations. However, studies have indicated that students may find proof and proving difficult to learn (Cabassut et al. 2012; Dreyfus, 1999; Küchemann & Hoyles, 2001a) and that teachers find it difficult to teach (Clausen-May, Jones, McLean, & Rowlands, 2000; Author, 2014; Author, 2008; Tatto et al., 2012).

To investigate why students have difficulties understanding, appreciating and formulating proofs, Harel and Sowder (1998) developed *proof schemes* to map students' cognitive schemes of proofs. Further research revealed that teachers' beliefs and behaviour might shape students' proof schemes (Harel & Sowder, 2007) and influence students' responses in geometric reasoning (Küchemann & Hoyles, 2001b; Diezmann, Watters, & English, 2002). Assuming that teachers' professional knowledge influences students' achievements (Hill, Rowan, & Ball, 2005; Tatto et al., 2012), effective teaching of geometric properties may result in effective learning and the development of deductive reasoning and analytical proof skills among students (Poon & Leung, 2016). Studies have shown that implementing preformal proofs in teaching practice may contribute to students' learning (Blum & Kirsch, 1991; Van Asch, 1993). Thus, teachers' knowledge regarding applying preformal proofs can enhance the effectiveness of their proof teaching skills, in which teacher education plays a critical role.

The employment of preformal proofs in mathematics lessons benefits not only students' learning and understanding but also teachers' professional development. Apart from knowledge of various mathematical theorems in the curriculum, Niss (2003), and Niss and Højgaard (2011) have indicated that a competent mathematics teacher should be able to excel in (i) mathematical thinking, (ii) posing and solving mathematical problems, (iii) mathematical modelling, (iv) mathematical reasoning, (v) mathematical representation, (vi) mathematical literacy and its formalism, (vii) mathematical communication, and (viii) mathematical aids and tools. Because they may demonstrate limited understanding of proof (Cabassut et al., 2012; Author, 2016; Knuth, 2002), teachers are encouraged to improve their mathematical competency through the utility of proofs in classrooms and simultaneously enrich their conceptions of proofs by implementing preformal proofs. It is expected that their mathematical and pedagogical knowledge can be strengthened.

In sum, we note a need for developing students' reasoning skills. By learning the proofs of geometric properties, students' deductive reasoning and analytical skills can be developed. We believe that teachers who implement mathematical learning through proofs may provide more opportunities and establish a suitable learning environment for students' exploration. If students have empirical difficulty in initiating a proof, teachers must use preformal proofs as an illustrative tool to scaffold initial ideas for formal derivations.

Theoretical Background

Preformal proofs versus formal proofs

Bruner (1966) noted that students' intelligence could be developed through the usual course from enactive through iconic to symbolic representation. The iconic representation reflects the intuitive idea, and symbolic representation constitutes the abstraction of such an idea. However, one's ability to represent a mathematics idea does not necessarily constitute mastery of the knowledge: this mastery involves deductive argument and logical derivation of mathematical concepts. In other words, this involves a leap from students' intuition of a mathematical proposition to be learnt to its mathematical justification, that is, a proof. According to Hilbert (1927, 1967), a mathematical proof—or a formal proof—is a sequence of well-organised assertions, in which each of the assertions is a logical axiom, a member of the sequence, or a result of applying the inference rule to some previous member. To connect students' understanding of a rough concept to a formal proof and to narrow the gap between students' intuitions and the formal proofs of this knowledge, an intermediate step, namely the *preformal proof*—a preliminary, logical, and insightful, but prematurely constructed idea or illustration that eventually hints to a formal proof—may be introduced. In analogue of Bruner's assertion, this preformal proof can be exhibited in the sequence of understanding an argumentation when proving a proposition or conjecture:

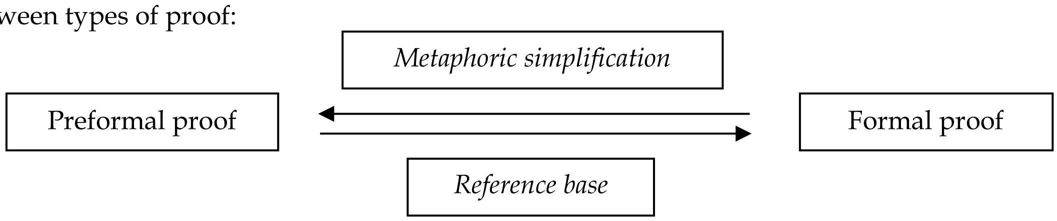
Enactive – Intuition – **Preformal proof ↔ Formal proof**

Here preformal and formal proofs are not necessary in a proper order of learning. Learners may think of the formal proof from intuition, and generate the preformal proof at the later stage of learning.

Lakatos (1978) introduced the term preformal proof. He distinguished between informal and formal proofs and further divided informal proofs into preformal and postformal proofs. Moreover, he revealed the importance of preformal proofs as the base and source for mathematicians in choosing the appropriate formal axioms to construct formal proofs (Bayat, 2016). Blum and Kirsch (1991) then systematically defined a preformal proof as 'a chain of correct, but not formally represented conclusions which refer to valid, non-formal premises' (p. 187). Preformal proofs are purposely presented in a manner that is 'as obvious and natural as possible' (p. 186). A preformal proof is a type of premature construction, rough sketch, draft tabulation, schematic imagery, or real object demonstration for establishing a relationship between variables. However, it is insufficiently mature to constitute a formal proof comprising logical arguments and reasoning. Reid (1993) applied the term preformal not only to proofs but also to proving (the process). He suggested that despite deficits in formal language and the explicit rules of inference, the reasoning behind preformal proofs fulfilled logical principles. More importantly, Reid indicated the necessity, but not the sufficiency, of a preformal proving ability as a prerequisite for formal proving ability.

Fujita, Jones, and Yamamoto (2004) asserted that to work on a proof exercise in geometry, students had to develop their geometrical intuition, which is 'a skill to create and manipulate geometrical figures in the mind, to see geometrical properties, to relate images to concepts and theorems in geometry, and decide where to start when solving problems in geometry' (p. 5). Preformal proofs can be generic, pragmatic, or visual proofs (e.g. proofs without words; Miller, 2012) and may be implemented as tools to assist students in understanding a proof and stimulating their cognitive activity and geometrical intuition by building their visual and empirical foundations for higher levels of geometric thought (Battista & Clements, 1995).

In mathematics learning, a preformal proof may serve as a stimulus or hint which eventfully leads to the complete structure of a formal proof. By contrast, a potentially effective construction of a formal proof requires a well-structured preformal proof. In other words, a preformal proof acts as a metaphoric simplification, a simplified illustration through a simple metaphor, of its corresponding formal proof. The following schematic diagram shows the interconnection between types of proof:



For example, a preformal proof of the triangle postulate involves tiling the interior angles of a triangle, which is a visual simplification of its corresponding formal proof and demonstrates the equality of the alternate angles under parallelism (Figure 1). By contrast, the formal proof provides a reference base to justify the preformal proof. In practice, a proficient teacher who can apply various preformal proofs may develop diverse approaches to formal proofs; however, a teacher who masters different formal proofs may simplify and modify them into various preformal proofs to aid students' learning. Figure 1 compares the corresponding steps in the two types of proof:

Step	Justification of the step	Claim of the step
1		In triangle ABC, define ∠BAC = α, ∠ABC = β and ∠ACB = γ.
2	Guaranteed by the 1st and 5th postulates of Euclid's Elements of Geometry.	Construct at A a parallel line to BC, namely, x′x where x′Ax is a straight line.
3		Define ∠x′AB = β′ and ∠CAx = γ′.
4	Definitions of alternate angles.	β and β′ are alternate angles and γ and γ′ are alternate angles.
5	Guaranteed by Proposition 29 of Euclid's Elements of Geometry.	β = β′ and γ = γ′
6	Definition of a straight angle.	∠x′Ax = 180°
7	Guaranteed by Proposition 13 of Euclid's Elements of Geometry.	∠x′Ax = ∠x′AB + ∠BAC + ∠CAx = β′ + α + γ′ = α + β′ + γ′
8	Calculus and deductive reasoning.	α + β + γ = 180°

Figure 1. Preformal and formal proofs of the triangle postulate.

Preformal proofs, visualisation and formal proofs

A generic preformal proof can help students understand and create proofs (Leron & Zaslavsky, 2013). When teaching the stated formal proof of the triangle postulate, teachers can provide its corresponding preformal proof (Figure 1) to demonstrate the equality of the alternate angles that leads students to derive the proof. In other words, given the statement of triangle postulate, the preformal proof provides a clue to students that the formal proof requires a partition (a tile of pieces adding up to form the whole) of the three interior angles, which eventually prompts the construction of a line parallel to a side of the triangle.

The potential contribution of visual imagery in mathematics learning has been debated for several decades (Dreyfus, Nardi, & Leikin, 2012; Nardi, 2014). For example, visualised preformal proofs are often criticised for being merely examples and therefore too specific to be used for general proofs (Nardi, 2014, p. 209). However, preformal proofs may be presented without loss of generality. Whiteley (2009) supported the visuality of preformal proofs because these can stimulate students' imagination of a range of variations of the figure by presenting a single image. Studies have also indicated that teachers and university students tended to agree that visual reasoning tasks can promote and enhance students' conceptual understanding, particularly for inexperienced learners and visual thinkers (Nardi, 2014; Natsheh & Karsenty 2014). In their study on 31 preservice elementary teachers, Stewart and Hadley (2014) stated that vivid visualisation abilities tended to positively correlate with pedagogical mathematics content knowledge in geometry; however, the correlation was nonsignificant.

Preformal proofs can provide students with less abstract and more directly linear logic by helping them grasp the concept through observation. However, Blum and Kirsch (1991) emphasised that preformal proofs are no simpler in terms of their references and lengths (step-size) than their corresponding formal proofs, especially if experienced mathematicians present a shorter and more sophisticated proof. Therefore, formal proofs remain essential in proving mathematical propositions and developing students' advanced deductive reasoning capability. This is echoed by Balacheff (2010):

> [The] answer to the question: "Can one learn mathematics without learning what a mathematical proof is and how to build one?" is "No" (p. 115).

Preformal and formal proofs in mathematics teacher education

Intuitively, presenting a preformal proof to students does not spontaneously help them to understand or create the corresponding proof. Teacher instruction is required to provide narrative arguments and explanations for students to build connections between preformal and formal proofs. Harel and Sowder (1998) stated that 'writing a mathematical argument appears to require instruction' (p. 278). Therefore, teachers should possess professional knowledge, such as proficiency of identifying different proof structures (preformal and formal), proficiency of knowing and executing alternative proofs, and the ability to recognise or establish connections between various mathematical knowledge domains (Author, 2008). By using preformal proofs, teachers can help students go beyond empirical proof schemes (Harel & Sowder, 1998, 2007) and enable them to write formal proofs with mathematical reasoning.

Teachers' beliefs and values regarding argumentation and proof also reflect and affect their competency and performance in classroom practice. Teachers who value the importance of argumentation and proof likely prepare more activities for students that provide experience of proofs and proving. In addition, teachers' beliefs influence their acceptance of different approaches to proofs. Biza, Nardi, and Zachariades (2009a, 2009b) indicated that certain teachers accepted a visual argument used for refuting a statement but not for proving it. Mathematical knowledge, pedagogical knowledge, and beliefs on argumentation and proof are interrelated.

To evaluate teachers' knowledge regarding proof teaching, Lin et al. (2012) proposed three central components: (i) teachers' knowledge of proofs (mathematical knowledge), (ii) the practice of proofs [pedagogical content knowledge (PCK)], and (iii) beliefs about proofs (views). In addition to general knowledge needed to teach mathematics, mathematics teachers should also acquire specific knowledge for teaching proofs (e.g. an understanding of the content of a proof, knowledge for justifying whether certain methods are valid, and the ability to validate students' proofs). Teachers' practice of proofs designates the type of proving activities that teachers implement in their classes to facilitate proof learning. In this study, we were concerned with teachers' views on proof and proving in their classroom instruction.

As an exploratory project, we were inspired by the aforementioned scholars' theories to investigate the preparedness of preservice and novice teachers (novice teachers are those without qualified teacher status who are teaching under the administration of school senior management; i.e. a nonlicenced teacher) for applying preformal proofs in mathematics teaching.

Research questions

Our study was focused on investigating preservice and novice teachers' views on the roles of preformal and formal proofs in geometry, with the triangle postulate as an example, and correlations between their views and backgrounds. This study explores the correlation between the implementation of preformal proofs and teachers' knowledge and beliefs on teaching proofs and proving. The findings can act as a reference for curriculum reform in mathematics teacher education. Based on the studies mentioned above, we formulated the following research questions within the prescribed scope:

1. How do preservice and novice teachers view preformal and formal proofs, given their knowledge of proofs, practice of proofs, and beliefs of proofs?
2. What are the features among teachers' backgrounds, particularly their mathematical and pedagogical knowledge and teaching experience, that may influence, affect, or dominate their preferences for preformal and formal proofs?

Methodology

Participants

In our study, the test instrument contained tasks regarding the preformal and formal proving of the triangle postulate. The questionnaires were delivered to 79 preservice and novice teachers, and 72 completed responses were returned. The respondents were students of teacher training programmes, in either their third or fourth year of undergraduate study in mathematics education or pursuing a postgraduate diploma in education with a major in mathematics. All respondents were from four universities in Hong Kong. The postgraduate respondents had already completed their undergraduate study, having majored in either mathematics or a mathematics-related subject such as engineering. A number of them were novice teachers or teaching assistants in secondary schools, and thus, they were enrolled in part-time programmes for a postgraduate diploma. For undergraduate and full-time postgraduate respondents, we expected that they would pursue a career in secondary school education after graduation. Table 1 presents the demographic variables. For a deeper understanding, we defined respondents' teaching experiences as classroom experiences such as regular lesson instruction and student teachers' practicums in mathematics at schools.

Table 1
Demographic variables of the respondents (n = 72)

	Undergraduate		Postgraduate		
	M	F	M	F	Total
Teaching exp.	9	4	20	10	43
No teaching exp.	10	3	13	3	29
Total	19	7	33	13	72

Remark: Teaching experience indicated teaching or practicum in a regular class at school.

The respondents were asked to complete a questionnaire with open-ended questions by commenting on three preformal proofs of the triangle postulate and were requested to formulate a corresponding formal proof. They were also asked to express their views and beliefs on proofs in secondary school mathematics. Containing six questions, the questionnaire covered several teachers' knowledge domains of preformal proofs and proving as outlined in the Theoretical Background section.

Questionnaire

Figure 2 depicts the task under evaluation. Before respondents responded to the questionnaire, three preformal proofs of the triangle postulate were introduced: preformal proof (a), presented in most of textbooks in France[1] as well as Hong Kong[2] as an exploratory in-class activity; preformal proof (b), sharing the concept proposed by Crowley (1987) regarding tessellations; and preformal proof (c), a paper-folding method described by Johnson (1999).

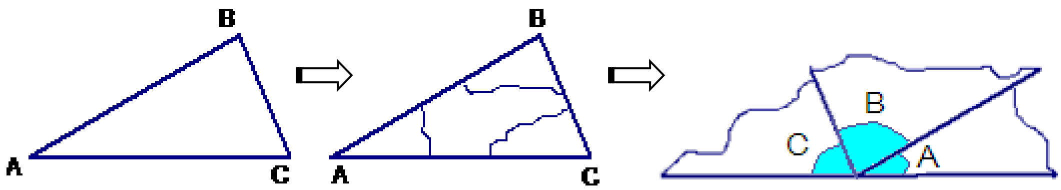
Item Set 3

Complementary approaches: formal – preformal proof about the angle sum of triangle.

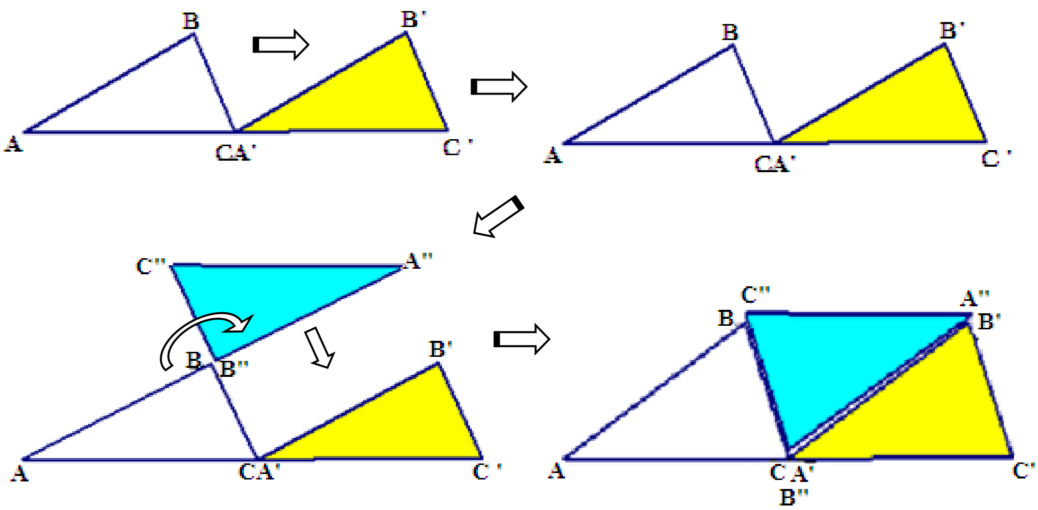
Consider the following three approaches. All three methods are called preformal proof to the proposition:

"Angles' sum of a triangle is equal to a straight angle".

(a) Tiling on the straight line – cutting the interior angles and tile them onto a horizontal straight line to make it a straight angle



(b) Tiling of triangles – duplicate a given triangle into two more identical triangles by various transformation: rotation and translation, then tile them onto a straight line: Vertices A′, B″ and C will meet at one point on the line AC′.



(c) Paper folding – Starting with a piece of triangular paper. A straight line is drawn from B to B′ on the base AC. The vertices A, B and C are folded respectively to meet at B′. The tree angles will then tile at B′ on the line AC.

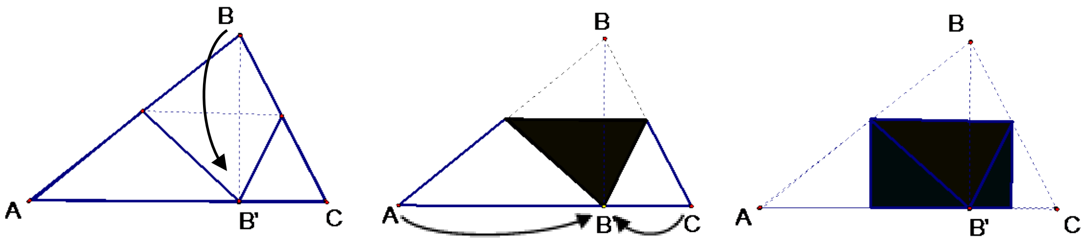


Figure 2. Item questions of preformal proofs of the triangle postulate.

These preformal proofs can be classified as action proofs because they involve placing all interior angles of a triangle on a straight line at a common vertex, resulting in a straight angle (Blum & Kirsch, 1991). Although the assumption that 'transformations such as reflection, rotation and translation, preserve the geometrical properties of a figure, in particular, the value of an angle' (Fischer, 2014) is shared by these preformal proofs, they involve different procedures that require various cognitive abilities, which may suit students at different learning stages. Specifically, preformal proofs (a) and (c) require students to observe the preservation of the angle size after tearing off or folding the corners; equivalently, students should have reached Piaget's stage of concrete operations, in which changing the shape of the figure does not change its measure (Kolb, 1984; Malerstein & Ahern, 1979). However, preformal proof (b) requires a visual understanding of congruent triangles and the concept of (equal-valued) corresponding angles. Through preformal proofs, teachers can enhance their pedagogical strategies by providing students with multiple activity-based learning opportunities for proofs to help students develop logical thinking and creativity in geometry (Arici & Aslan-Tutak, 2015; Bostic, 2016; Coad, 2006; Levav-Waynberg & Leikin, 2012; Ojose, 2008).

Regarding the preformal proofs in Figure 2, the respondents were asked the following questions, based on those of Author (2008):

Q1. Which of the above methods/approaches is (are) sufficient proof(s) for you? (You accept that it is a proper proof of the proposition). Please give a short explanation.
Q2. Formulate a formal proof for the proposition.
Q3. Which kind of proofs (between preformal proofs ((a), (b), (c)) and formal proof you showed in Q2) would you use in your mathematics lesson? Please explain your position.
Q4. In a mathematics lesson, do you think preformal proof (any one of the above three methods) is sufficient? Please explain.
Q5. Name the advantages and disadvantage of preformal and formal proof.
Q6. How do you think about the meaning of mathematical proofs in a lesson?

Data analysis

The analytical method used to analyse the written responses was adapted from the deductive category application designed by Mayring (2000) and modified by Author (2008). Several criteria were predefined before data analysis to devise a specific structure. A coding manual, comprising definitions, typical examples of responses, and coding rules, was then constructed to analyse and code the data (i.e. to assign the data according to the predefined evaluation categories). Quantitative analyses were subsequently conducted according to the frequency.

Analysis of Responses

The foci of this study were teachers' knowledge of proofs, practice of proving, and beliefs about proofs. Respondents' views on preformal proofs were analysed by considering (i) their acceptance of the presented preformal proofs (i.e. the validity of the preformal proofs) and (ii) their preferred types of proofs in a classroom setting (i.e. practice of proofs). Regarding the reasons for consideration, their responses were further subcategorised as (a) concerns of student learning, (b) practical concerns, (c) mathematical content concerns of the proofs, (d) concerns of the natures of preformal proofs (e.g. visualisation and pragmatics), and (e) concerns of students' attitudes (affection). Frequencies of the categorised considerations were counted to examine the tendencies of considerations when the respondents reflected on teaching and learning preformal proofs (Appendix 1).

Moreover, respondents' views on proofs (beliefs about proofs) were examined by categorising their reflections on the meaning of proofs in a classroom setting, regarding whether they (a) consider proofs as a justification to convince students, (b) appreciate the nature of proofs as a key element of learning mathematics, (c) consider that learning proofs can help students' cognitive development and logical reasoning, (d) consider that learning proofs can consolidate student mathematical learning and understanding of certain topics, (e) consider that learning proofs is a supplement to the school curriculum, and (f) deny any advantage of teaching and learning proofs (Appendix 1).

Scales for Respondents' Knowledge of Proofs, Practices of Proofs, and Beliefs about Proofs
To investigate teachers' knowledge of proofs by restricting our analysis to school mathematics, we developed a coding manual focusing on teachers' abilities (i) to identify both differences and similarities between preformal and formal proofs, (ii) to justify the validity of the proposed preformal proofs, and (iii) to develop a formal proof by referencing the preformal proofs presented. An ordinal scale was also constructed, ranging from 0 (respondent does not demonstrate any above-mentioned ability while responding to the items) to 6 (respondent demonstrates all abilities; Appendix 2).

For teachers' practice of proofs, regarding the implementation of preformal proofs in class settings, we constructed another ordinal scale to examine (i) teachers' preference for proving activities and (ii) their acknowledgement that implementation of preformal proofs can benefit student learning. A respondent could achieve a score of 8 by adapting all preformal and formal proofs to mixed teaching strategies. The scale also identified teachers' pedagogical knowledge about preformal proofs: (a) preformal proofs can provide opportunities for visual and pragmatic learning, (b) preformal proofs can cater to inexperienced or less capable students, and (c) preformal proofs can serve as a stepping stone to facilitate the development of students' higher-order thinking skills (Appendix 2).

Because of the affective feature of teachers' beliefs about proofs and the cognitive components of beliefs, an ordinal scale was defined to indicate the degree of their agreement and refusal with teaching mathematical proof and proving. The scale ranges from −2 (respondent refuses to teach proofs or denies any positive impacts of learning proofs on student development) to +2 (respondent advocates teaching and learning proofs; Appendix 2).

In our study, information on the respondents' academic backgrounds was also collected to examine the pairwise correlations between their backgrounds, knowledge of proofs, practice of proofs, and beliefs about proofs. Respondents' academic backgrounds included (a) mathematics courses and (b) mathematics education courses (Appendix 3).

Results and Findings

Knowledge of proofs and practices of proofs

The structural associations between the different components of novice and preservice teachers' knowledge in Hong Kong are described as follows: Q1 aimed to examine the respondents' judgements on the acceptance of the three preformal proofs of the triangle postulate (general results in Table 2). All methods received approximately half of the respondents' acceptance and yielded similar results overall. The chi-square tests of independence were performed to examine the pairwise relationships among the acceptances of the three approaches to preformal proofs. A significant relationship was noted between the acceptances of preformal proofs (a) and (b) ($\chi^2 = 32.56$, $p < 0.001$), (a) and (c) ($\chi^2 = 22.77$, $p < 0.001$), and (b) and (c) ($\chi^2 = 22.77$, $p < 0.001$). Thus, the preservice teachers who accepted any one of three preformal proofs tended to accept the others as well.

Table 2
Acceptances of three approaches to preformal proofs (frequency counts out of n = 72)

	Accept	Reject	Blank/Indecision
(a) Tiling on straight line	37 (51.4%)	13 (18.1%)	22 (30.6%)
(b) Tiling of triangles	36 (50.0%)	13 (18.1%)	23 (31.9%)
(c) Paper folding	38 (52.8%)	13 (18.1%)	21 (29.2%)

Table 3 lists the reasoning behind the responses. For thematic concerns, the responses were divided into six categories regarding (a) student learning, (b) practical concerns, (c) mathematical content of the proofs, (d) physical natures of the proofs (e.g. visualisation and pragmatics), (e) students' attitudes towards learning proofs and mathematics (affection), and (f) other or unrelated concerns.

The results indicated that the respondents tended to accept the preformal proofs because they can visually or pragmatically demonstrate the proposition of the triangle postulate, but they tend to consider practical issues (e.g. overlapping when placing three angles together) when rejecting the proofs. In addition, several respondents denied the generalisability of the preformal proofs. Table 3 shows respondents' considerations.

> None of them is a sufficient proof for me. They only show that some Δ [exists, and] the angles sum = 180°. They are [in]sufficient to prove that all Δs' angle sum = 180°. [#48, coded mathematical content concerns of the proofs (M).]

> As a secondary school teacher, or a university student, none of the above method[s] is sufficient. However, [from] secondary school student's perspective, all of them are intuitively acceptable [because] their rigor[ous] reasoning ability [has] not [been] well-developed yet. [#65, coded concerns of student learning (L), and physical natures of the proofs (PN).]

Table 3
Reasons for acceptances or rejections of three approaches to preformal proofs

	SL	P	MC	PN	A	Other/unrelated
(a) Tiling on straight line						
Accept	8	4	4	28	2	2
Reject	0	7	5	1	0	1
(b) Tiling of triangles						
Accept	4	4	14	22	3	1
Reject	1	1	4	1	0	1
(c) Paper folding						
Accept	6	5	8	23	2	1
Reject	0	8	4	1	0	0

Q2 did not require advanced mathematical knowledge, such as a thorough understanding of proofs from undergraduate to advanced levels, but it required a type of knowledge called 'mathematics for high schools from an advanced perspective' (Usiskin, Peressini, Marchisotto, & Stanley, 2003; Buchholtz et al. 2013). For Q2, the preservice teachers were asked to formulate a formal proof for the following mathematical proposition: '(Interior) angles' sum of a triangle is equal to a straight line' (i.e. the triangle postulate), which is Proposition 1.32 in Euclid's *Elements* (Fitzpatrick, 2007, p. 34). To complete this task adequately, elementary geometrical propositions, namely Propositions 1.13, 1.29, and 1.31 in Euclid's *Elements* (Fitzpatrick, 2007, pp. 34–

35)[3] and competencies in the systematic formulation of a formal proof in mathematical knowledge, are required.

Table 4
Approaches of formal proof formulation (Q2)

Attempts (*n* = 65)	Visually assisted	Deductive proof	Incomplete proof	Unsuccessful
Frequency	53	29	14	22
% out of attempts	81.5%	44.6%	21.5%	33.8%

Because some respondents misinterpreted the task or left a blank space, 65 valid attempts (90.3% of the 72 respondents) were considered. The results that 44.6% of the participants could formulate a deductive proof corroborated the study by Author (2008), in which 42.7% of the preservice teachers from Hong Kong could formulate a formal proof for the proposition in the task (i.e. coded +1 or +2 responses; ibid., p. 798). Moreover, 53 respondents attempted this task by sketching a figure as a visual assistance or illustration. Figure 3 shows example responses.

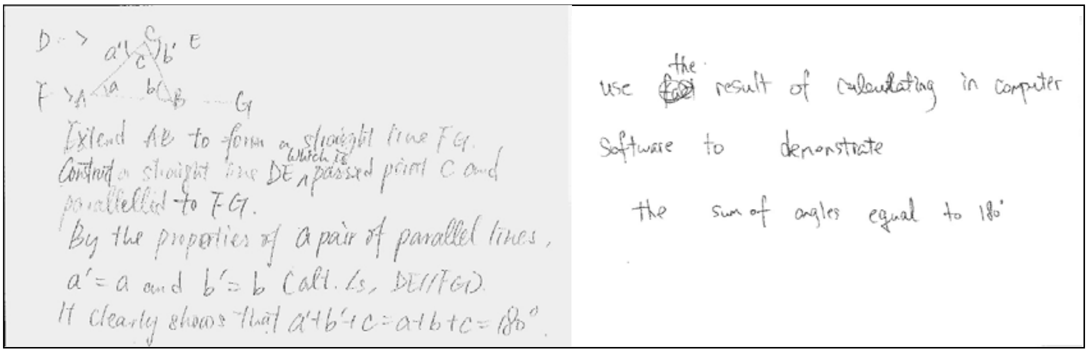


Figure 3. Example responses to Q2 (left: a deductive proof; right: an unsuccessful attempt).

In addition, qualitative content analysis was conducted to determine the preservice teachers' preferred approaches to formulating a formal proof of the triangle postulate. Of the 65 respondents, 42 (64.6%) attempted to formulate a proof by constructing a parallel line to one side of the triangle passing through the remaining point and applying the properties of the angles related to parallel lines. However, three respondents attempted to apply the proposition 'The sum of the exterior angles of a convex polygon is a round angle (four right angles)' to formally prove the triangle postulate, and one could provide a brief justification for using the concept of limit to validate this proposition prior to applying it. Here, the use of the concept of limit reflects that a formal proof of the triangle postulate is also a consequence of tiling the exterior angles at a singleton point (Figure 4).

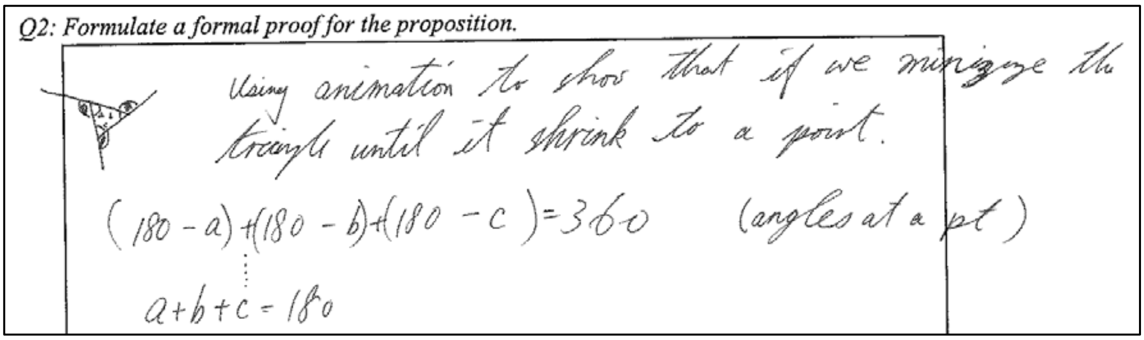


Figure 4. Example response to Q2 (incomplete proof), the property of exterior angles is used.

Among the 22 unsuccessful attempts, six simply restated one of the preformal proofs in the questionnaire without justification, three involved circular arguments, five attempted to verify or measure the sum of the angles of particular triangles, six left spurious or irrelevant comments, including 'no formal proof', 'I don't know', and 'I forget' and the remaining identified incorrect pairs of alternate angles of parallel lines. These responses revealed the respondents' below-average competencies in knowledge of proofs with respect to understanding what constitutes a proof, specifically the deduction of a formal proof and its formulation.

The results for Q3 and Q4 were related to respondents' practice of proofs, specifically targeting preformal proofs and proving, and their connection with formal proofs in classroom settings. Notably, because respondents were allowed to provide reasons with various considerations (different reasons of the same type of consideration were counted as a single type), the number of reasons exceeded 72 (the total number of respondents). In addition, blank responses were not considered.

In Q3, respondents were asked to provide their preferred presented proof and teaching strategies for teaching the triangle postulate in a classroom setting. Table 5 shows respondents' preferred proofs in terms of their demographics. The results indicated that the majority of respondents (58.3%) selected preformal proof (a) for their triangle postulate teaching strategy. Furthermore, despite visible differences between percentages, no significant relationship was found between respondents' preferences for formal proof and their teaching experience (χ^2 = 2.04, p = 0.154), their preference for preformal proof (c) and their teaching experience (χ^2 = 1.22, p = 0.269), or their preference for preformal proof (c) and their education (χ^2 = 1.05, p = 0.305). Three additional responses were found (coded 'other'), two of which involved refusal to teach proofs and one stated a preference to measuring angles.

Table 5
Respondents' preferences for specific proofs in classroom settings

	(a) Tiling on straight line	(b) Tiling of triangles	(c) Paper folding	(d) Formal proof	(e) Other
Postgrad	26 (56.5%)	13 (28.3%)	19 (41.3%)	17 (37.0%)	1 (2.2%)
Undergrad	16 (61.5%)	10 (38.5%)	14 (53.8%)	10 (38.5%)	2 (7.7%)
Teaching exp.	23 (53.5%)	13 (30.2%)	22 (51.2%)	19 (44.2%)	3 (7.0%)
No teaching exp.	19 (65.5%)	10 (34.5%)	11 (37.9%)	8 (27.6%)	0 (0.0%)
Total	42 (58.3%)	23 (31.9%)	33 (45.8%)	27 (37.5%)	3 (4.2%)

Remark: The percentages are based on the categories (e.g. Postgrad n = 46, Table 1).

For Q4, the respondents were asked to justify the sufficiency of using preformal proofs alone in classroom settings. Table 6 reveals that most respondents (63.9%) agreed that preformal proofs can be used alone for teaching the triangle postulate, whereas the remainder insisted on teaching formal proofs because of the lack of rigor provided by preformal proofs for student's cognitive development. In addition, one respondent claimed, 'I have never thought about it'. Furthermore, despite a visible difference, the relationship between respondents' acceptance of using preformal proofs alone in classroom settings and their teaching experience was nonsignificant ($\chi^2 = 1.53$, $p = 0.216$).

> Lack of logical thinking is a flaw concerning advanced mathematical education. [#77]
>
> Via preformal proofs, abstract theorems are more acceptable to students. However, mathematically speaking, rigorous proofs and algebraic deduction are important. [#79]

Table 6
Respondents' views on using preformal proofs alone in classroom settings

	Accept	Reject
Postgrad	29 (63.0%)	17 (37.0%)
Undergrad	17 (65.4%)	9 (34.6%)
Teaching exp.	25 (58.1%)	18 (41.9%)
No teaching exp.	21 (72.4%)	8 (27.6%)
Total	47 (63.9%)	25 (36.1%)

Remark: The percentages are based on the categories (e.g. Postgrad $n = 46$, Table 1).

On combining the responses of Q3 and Q4, we noted that 23 respondents (31.9%) indicated that they would teach the triangle postulate along with the mixed strategies of adapting both preformal and formal proofs, whereas 20 indicated that they would implement a strategy of 'first preformal proof(s), then formal proof(s)'. (Table 7) Furthermore, the adaptation of both preformal and formal proofs tended to correlate with respondents' teaching experience ($r = 0.259$, $p < 0.05$; Table 11).

Table 7
Respondents' strategies of teaching the triangle postulate

	Preformal first	Formal first	Undecided	Sum
Postgrad	13 (28.3%)	1 (2.2%)	1 (2.2%)	15 (32.6%)
Undergrad	7 (26.9%)	0 (0.0%)	1 (3.8%)	8 (30.8%)
Teaching exp.	16 (37.2%)	0 (0.0%)	2 (4.7%)	18 (41.9%)
No teaching exp.	4 (13.8%)	1 (3.4%)	0 (0.0%)	5 (17.2%)
Total	20 (27.8%)	1 (1.4%)	2 (2.8%)	23 (31.9%)

Remark: The percentages are based on the categories (e.g. Postgrad $n = 46$, Table 1).

In addition, to perform a holistic investigation, respondents' considerations for responses among the first five items (Q1–5) were analysed. The reasons were categorised as (a) student learning, (b) practical concerns, (c) mathematical content (particularly respondents' rejection of the generalisation power of preformal proofs), and (d) students' attitudes towards learning proofs and mathematics (affection). All responses could be categorised into at least one of the above categories. Thus, no other category was required. Despite nonsignificant results ($p \geq 0.05$), an association was revealed between respondents' teaching experience and responses of practical concerns ($r = 0.200$, $p = 0.0917$; Table 11). Furthermore, we found that 20 respondents (27.8%) rejected or doubted the generalisation power of the preformal proofs.

> Q1: I may wonder [whether] only this triangle [works], how about [the] other[s]? Q3: (a) since formal proof may not [be understood by] all students, (b) need[s] more resource[s], (c) may not apply for all types of triangles, just like (a right triangle). Q4: Yes, some concept[s] [are] only suitable [for observation] by students. Formal proof[s] may be too difficult for them to understand. [#25, coded concerns of student learning (SL), practical concerns (P), mathematical content concerns (MC), and the rejection of the generalisation power of preformal proofs.]

Table 8

Considerations for responses among Q1–5

	SL	P	MC	*Reject gen.*	A
Postgrad	45 (97.8%)	31 (67.4%)	42 (91.3%)	14 (30.4%)	17 (37.0%)
Undergrad	23 (88.5%)	17 (65.4%)	21 (80.8%)	6 (23.1%)	10 (38.5%)
Teaching exp.	40 (93.0%)	32 (74.4%)	38 (88.4%)	11 (25.6%)	14 (32.6%)
No teaching exp.	28 (96.6%)	16 (55.2%)	25 (86.2%)	9 (31.0%)	13 (44.8%)
Total	68 (94.4%)	48 (66.7%)	63 (87.5%)	20 (27.8%)	27 (37.5%)

Remark: The percentages are based on the categories (e.g. Postgrad $n = 46$, Table 1).

Beliefs about proofs

For the qualitative approach, respondents' views on proofs are listed in Table 9. Thirty respondents (41.7%) stated that learning proofs can consolidate student learning of the target topic, whereas three respondents denied any benefit from learning proofs. Furthermore, Respondents without teaching experience tended to express more appreciation for the natures of proofs ($r = -0.249$, $p < 0.05$).

> [Learning proofs is] not much useful. Whenever my teachers taught this, we did not listen. I do not believe that students will listen to the explanation of proofs in the lessons. [#58, coded denying of benefit from learning proofs (N).]

> In junior form, proofs and explanations should be easier, in order to make the students to grasp the concept clearly and not to ruin their interest and sense of failure in maths; in senior form, the students become more mature, so we can explain the maths content in a deeper way so that we can train their logical reasoning and thinking. [#21, coded appreciating the nature of proofs (A), proof for developing students' cognition and logical thinking (D), and proof for consolidating student learning (C).]

Table 9

Respondents' views on proofs in six categories

	J	A	D	C	S	N
Postgrad	19 (41.3%)	15 (32.6%)	13 (28.3%)	16 (34.8%)	5 (10.9%)	2 (4.3%)
Undergrad	8 (30.8%)	5 (19.2%)	5 (19.2%)	14 (53.8%)	1 (3.8%)	1 (3.8%)
Teaching exp.	16 (37.2%)	8 (18.6%)	12 (27.9%)	17 (39.5%)	5 (11.6%)	3 (7.0%)
No teaching exp.	11 (37.9%)	12 (41.4%)	6 (20.7%)	13 (44.8%)	1 (3.4%)	0 (0.0%)
Total	27 (37.5%)	20 (27.8%)	18 (25.0%)	30 (41.7%)	6 (8.3%)	3 (4.2%)

Remark: The percentages are based on the categories (e.g. Postgrad $n = 46$, Table 1).

Scales for knowledge of proofs, practice of proofs, and beliefs about proofs

In addition to the qualitative approach, respondents' knowledge of proofs and practice of proofs were examined by a construct of ordinal scales (quantitative approach). For knowledge of proofs, a 7-point scale (0–6) was developed regarding (a) respondents' acceptance of each preformal proof, (b) their abilities to formulate a deductive proof of the triangle postulate with or without reference to any presented preformal proof, and (c) their knowledge of the connection between preformal and formal proofs. Respondents scored 6 if they acknowledged that all presented preformal proofs were valid and could formulate a deductive proof of the triangle postulate with recognition of the connections between preformal and formal proofs. The average score was 2.63 [standard deviation (SD) = 1.50].

Moreover, for the practice of proofs, a 9-point scale (0–8) was constructed regarding (a) respondents' preference for each type of presented proofs (both preformal and formal), (b) their preference for a mixed strategy of preformal and formal proofs for teaching the triangle postulate, (c) understanding that a preformal proof can provide students with visual or pragmatic learning opportunities, (d) understanding that a preformal proof can help less capable students learning, and (e) understanding that a preformal proof can act as a stepping stone for students to advance a higher-order thinking, particularly to facilitate logical thinking skills. Respondents scored 8 if they demonstrated a holistic teaching strategy including all presented proofs and acknowledged a range of benefits of learning (preformal) proofs. The average score was 3.01 (SD = 1.98).

Assuming that the respondents answered the items consistently in the questionnaire, we investigated respondents' beliefs about proofs regarding preformal and formal proofs by evaluating their responses to Q6 on a 5-point integral scale (−2 to +2). An elaborate positive statement for proof and proving with at least two aspects regarding (a) mathematical considerations and (b) didactical considerations was coded +2 (advocating proofs) as well as a positive statement containing one of these aspects was coded +1 (tendency of proofs). A statement revealing a tendency for degrading proof teaching and learning was coded −1 (tendency of rejecting proofs), and one denying any positive meanings of proof teaching and learning was coded −2 (rejecting proofs). Any indecisive statement, which may be unjustified, incomprehensible, or lacking clarity in meaning, was coded 0 (neutral). The average score was 0.458 (SD = 0.897).

> [The presence of proofs in the curriculum] is [to] tell them where the mathematics theorems [are derived] from, [to] develop their mathematics thinking. [to open] their eyes to some abstract maths proofs, [and] to show them [that] much effort is paid to [devising] such proofs. [#32, coded +2.]

> At the secondary level, mathematical proofs are complementary to student learning. Sometimes it is not necessary for a mathematical proof to really help students understand the topic. [#47, coded −1.]

Pairwise Correlations (r) between Respondents' Knowledge, Practices, and Beliefs about Proofs

For deeper investigation, respondents' views on proofs were examined regarding (a) their knowledge of proofs, (b) their practice of proofs, and (c) their beliefs about proofs. Pairwise correlations (r) between the three aspects, and between their components were investigated (Table 10). The results indicated moderate positive correlations among the three aspects ($r_{\text{KoP,PoP}} = 0.423$, $p < 0.001$; $r_{\text{KoP,BaP}} = 0.308$, $p < 0.01$; $r_{\text{PoP,BaP}} = 0.385$, $p < 0.001$); therefore, a respondent achieving high score in one scale, tended to achieve high scores in the other two. Furthermore, respondents tended to implement a mixed strategy of preformal and formal proofs to teach the triangle postulate if they had solid knowledge of proofs ($r_{\text{KoP,mixed}} = 0.282$, $p < 0.05$). They acknowledged that adapting preformal proofs in classroom settings can provide students with visual and pragmatic learning opportunities ($r_{mixed,VP} = 0.241$, $p < 0.05$), and believed that preformal proofs can foster the development of student's higher order thinking skills ($r_{mixed,HOT} = 0.379$, $p < 0.01$) (see Table 10).

Table 10
Pairwise correlations (r) between respondents' knowledge of proofs, practice of proofs and beliefs about proofs

	KoP	*Form P*	*Connect*	PoP	*Mixed*	*VP learn*	*Less cap.*	*HOT*	BaP
KoP									
Form P									
Connect		0.251*							
PoP	0.423***	0.353**	0.212+						
Mixed	0.282*	0.305**	0.134						
VP learn	0.057	0.190	0.163		0.241*				
Less cap.	0.066	0.074	0.009		0.176	0.009			
HOT	0.236*	0.078	0.252*		0.379**	0.220+	0.356**		
BaP	0.308**	0.194	0.212+	0.385***	0.177	0.213+	0.223+	0.231+	

Remark: + $p < 0.10$. * $p < 0.05$. ** $p < 0.01$. *** $p < 0.001$.

Pairwise Correlations between Respondents' Backgrounds and Their Views on Proofs

Moreover, respondents' backgrounds were examined regarding (a) their teaching experience, and (b) their education, namely (i) mathematics courses and (ii) mathematics education courses. Pairwise correlations between these components were investigated (Table 11). Respondents' teaching experience tended to influence their choices of teaching strategies ($r_{TE,mixed} = 0.269$, $p < 0.05$), their concerns about teaching and learning proofs ($r_{TE,VP} = 0.293$, $p < 0.05$; $r_{TE,HOT} = 0.219$, $p < 0.10$), and their practice of proofs ($r_{TE,PoP} = 0.235$, $p < 0.05$). Furthermore, the results of pairwise correlations also indicated that respondents' knowledge of proofs, practice of proofs, and beliefs about proofs may be influenced by their knowledge of mathematics and mathematics education, which is discussed in the following section (see Table 11).

Table 11
Pairwise correlations (r) between respondents' backgrounds and their views on proofs

		Mathematics					Mathematics education							
	Teaching exp.	NT	Geo	Algeb	Prob & Stat	Math analy	No. of courses	Curr	Prob solving	Math history	ICT	Psych	Acti & games	PCK
KoP					-0.287*					0.232		0.206		
Form P										0.201				0.208
Connect			0.233*						-0.259*					
PoP	0.235*		0.220	-0.241*			0.217				0.307**			
Mixed	0.269*			-0.324**			0.254*				0.287*	0.367**		0.211
VP learning	0.293*													
Less cap.					-0.282*									
HOT	0.219								-0.218		0.303**		0.250*	
BaP					-0.226									
Concerns														
SL														
P										-0.216				
MC		0.274*						0.237*						
A						0.279*								

Remark: Only correlations *r* with $p < 0.10$ are presented. * $p < 0.05$. ** $p < 0.01$.

Summary of the findings

In this section, we summarise our findings according to the responses of all questionnaires. In Q1, although we expected a competent mathematics teacher to assess the sufficiency of a preformal proof mainly on the basis of its mathematical contents, such as original definition, features, principles, properties, and implications, fewer than a half of the respondents were found to accept or reject a proof 'mathematically' (Table 3). The choice of acceptance reflects their tendency to use preformal proofs under didactical considerations.

In Q2, a high proportion of respondents (81.5%) generated a figure when attempting to formulate a deductive proof for the triangle postulate. This indicated respondents' dependence on proving geometric propositions through a static figure of objects, whereas formal proofs require comprehensive and precise definitions of the axiomatic objects. In addition to learning from the preformal proofs presented, a probable explanation of the predominance of the strategy of constructing parallel lines (64.6%) is that such strategy is often described and taught in Hong Kong textbooks under the topic of deductive geometry. Thus, further and deeper investigations are required to examine respondents' ability of deductive proof formulation by referencing a preformal proof; in other words, whether the respondents have already learnt and adopted the proof from some textbook or developed the deduction themselves with reference to the presented preformal proof.

In addition, approximately one-fourth (27.8%) of the respondents did not acknowledge that the preformal proofs may stimulate students to imagine a range of similar figures for generalisation but preferred to use a particular example. Furthermore, combined with the diverse data regarding knowledge of proofs (M = 2.63, SD = 1.50), the results suggested a need for improvement of knowledge of proofs among preservice and novice teachers. Moderate correlations between knowledge of mathematics history and the scale of knowledge of proofs may indicate appropriate adjustments to teacher education programmes.

Moreover, respondents' practices of proofs were examined. We found that although all preformal proofs presented here were equally accepted, preformal proof (a) dominated the respondents' preferences (F = 1.255, p < 0.01) because of its frequent appearance in Hong Kong textbooks. Moreover, Table 11 indicates that respondents who acknowledged the use of information and communication technologies (ICT) in mathematics teaching tended to demonstrate various teaching strategies and acknowledge the advantages of implementing preformal proofs in classroom settings. Acknowledging some investigations on the relationship between student learning and mathematics textbooks (Kwong, 2013; Leung, 2005), the relationship between teachers' instruction and their use of textbooks and ICT tools in proof teaching should be investigated. Teachers' knowledge and preferences would affect their choices of learning resources such as textbooks and ICT tools; however, the availability of developed learning resources may clarify new teaching strategies to cater for learning diversity.

Discussion

Relations among beliefs, practice, and knowledge about proof teaching

Figure 5 presents a schematic of the connections between the implementation of proof teaching, both preformal and formal proofs, in the classroom. The arrows denote the sequential directions of execution, in which knowledge of proofs is the foundation. The ability to interconnect the preformal and formal, as described in the Theoretical Background section, reflects teachers' competency. This ability serves as a foundation for a strong belief in, and good practice for, teaching proofs. Good practice and strong positive belief are interdependent.

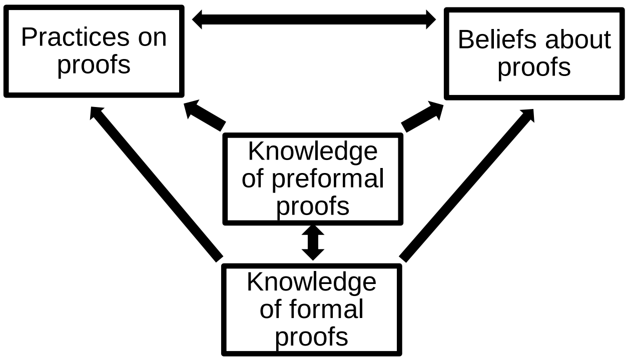


Figure 5. Schematic implementation of preformal and formal proofs.

The aim of this investigation was to assess teachers' proficiency and affinity for various preformal proofs of the triangle postulate. In the mathematics curriculum of Hong Kong, the application of the Euclid's parallel postulate is a core content of the lower secondary syllabus, evidenced by its statement and frequent appearance in the deductive proofs of various geometric properties. However, its equivalence with the triangle postulate is not emphasised. The textbook demonstrates a deduction of the triangle postulate, represented by the proposition: *the sum of interior angles of a tringle is 180°*, from the truth of parallel postulate, but omits the equivalence of the two postulates.

The approach that applied angle tiling [either exterior (Figure 4) or interior angle] is an essential idea of preformal proofs. In the questionnaire, preformal proof (a) is straightforward, whereas preformal proofs (b) and (c) require some applications of geometric properties—linear transformations of a triangle in preformal proof (b) and the midpoint theorem of a triangle in preformal proof (c). Applying these two properties leads to the tiling of the three (interior) angles, which physically illustrates the action of adding the angles' measures. We expect that the participant teachers could show this step to students because the proposition requires showing the property of the sum of interior angles. This sum prompts the teachers and students to think of putting them together on a plane, which is simply a tessellation at a point.

A preformal proof is not a synonym for an example. The tiling of the angles of a paper-made triangle builds an image in learners' memory that the choice of the triangle, hence, the three interior angles, is arbitrary. The triangle is chosen without loss of generality because the three angles' measures do not affect the result of tiling and are not known in the proof. In other words, the triangle should be chosen arbitrarily in order for the construction of a preformal proof. Otherwise, the combination of fixed measures, for example, 30°, 60°, and 90°, is only a demonstration, an illustrative example, or merely a verification.

Tiling three interior angles at a point on a straight line hints at the construction of a line parallel to the opposite base of the triangle. The participating preservice and novice teachers were provided with the idea of constructing a parallel line to formulate a formal proof when reading the picture illustration of the angles tiling in the three preformal proofs in the questionnaire. To assess teachers' ability to use this approach in teaching a proof of geometric proposition, we argued that a qualified mathematics teacher should know the complete process of a formal proof of the triangle postulate because it is quite a standard foundation in Euclidean geometry. Moreover, a competent teacher is expected to be able to simplify the formal proof and mediate it to create a preformal proof for student learning. They can eventually lead students to arrive from deductive arguments to the result of the triangle postulate, which represents proficiency in interconnecting preformal and formal proofs.

'Competency is an integrated whole of knowledge, insights, skills, and attitudes' (Vermunt, 2007, p. 84). A mathematically competent teacher can insightfully think of novel approaches or aids to deal with challenges in teaching. The three preformal proofs stated in the questionnaire already hinted at the construction of a parallel line in our suggested solution of proving the triangle postulate. The physical cut-and-tile of the three interior angles of a triangle in those preformal proofs prompted their geometrical intuition (Fujita et al., 2004). From this, the participating teachers should obtain insights into constructing a line passing through the vertex parallel to the opposite base. To facilitate students' understanding of core content for those with different learning needs, a competent teacher should obtain the insight and have the capacity to use aids (Niss & Højgaard, 2011). The construction of the parallel line in our case particularly requires teachers' insight. Competent teachers do not need to recall memory on the construction but obtain the insight from the preformal proofs.

Teachers can make effort to prepare for a good lesson for teaching proofs. We expect that respondents should have learnt the formal proof of the triangle postulate indirectly by reading the three methods of preformal proofs demonstrated. A competent teacher can interconnect the approaches of the two types of proofs, preformal and formal, with some of the hints provided.

Contingency is one of the teachers' knowledge competencies described in Rowland's *knowledge quartet* mathematics teaching model (Rowland, Huckstep, & Thwaites, 2005; Rowland & Turner, 2007). A competent teacher can create a formal proof in a lesson (described in this paper) without prior preparation when students ask for a formal, deductive formulation of the proof after the teacher has demonstrated the preformal proofs. A competent teacher with rich contingency can respond to irregular and unpredictable scenarios during teaching.

Conclusion

Affinity with using preformal proofs

Competent mathematics teachers are commonly believed to possess many different approaches in delivering abstract mathematical concepts or ideas. Lower secondary mathematics students, who encounter mathematics content including justification and logical arguments, may experience frustration regarding how to start a proof. Teachers' ability to illustrate a preformal proof of the triangle postulate helps students to grasp the idea of adding the angles, tiling them through transformation, and eventually constructing the formal proof.

Our study reveals a problem encountered during mathematics teacher education: Mathematics educators emphasise the PCK of teachers, and they are expected to be capable of teaching mathematical proofs. However, the local syllabi of teacher education colleges skew the focus on teachers' PCK and lack elaboration of interconnecting PCK with pure subject knowledge in terms of proof teaching and information that a proof requires rich subject content knowledge through axiomatic, deductive reasoning. Although proficiency of using preformal proofs reflects teachers' PCK, Barlow and Reddish (2006) asserted that mathematical ideas are initially generated in intuitive notions, such as preformal proofs in this investigation, and that deduction is an essential skill for mathematical argument and formal proof. Competency in subject knowledge is the foundation to help teachers create a suitable preformal proof for students' mathematical interpretation (Ball et al., 2008). In this study, we found that the participating teachers were quite comfortable using preformal proofs in lessons given their belief that students can easily grasp the idea about the property of the angle sum of a triangle.

Study limitations

Preservice and novice teachers are unfamiliar with the term preformal proof because it never appears in local mathematics syllabi. However, we lack sufficient evidence to determine whether teachers are generally proficient in using preformal proofs in other topics of mathematics teaching.

Although we believe that the angle-tiling procedure (preformal proof (a)) is illustrated in most local textbooks, teachers probably treat it as an alternative proof for students to easily understand the triangle postulate. This study contained only 72 samples; we simply diagnosed teachers' knowledge on using preformal proofs and their preference for proof teaching in a hypothetical setting. For a complete understanding of how well prepared our mathematics teachers are in terms of the proficiency and knowledge competency of teaching proofs, we believe that a comprehensive investigation into teachers' knowledge of proofs in various topics (e.g. algebra and geometry), with different scopes, such as calculus and set theory, is necessary. Our investigation only aimed to test teachers' knowledge of preformal proofs and whether they preferred to use them in classroom instruction of the triangle postulate. Moreover, the comparisons between in-service and preservice teachers on the use of preformal proofs and proof teaching can also be studied to improve mathematics teachers' knowledge of proofs and the teaching of proofs.

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Appendices

Appendix 1

Table 12
Reason for acceptances or rejections of preformal proofs

Code	Definition	Coding rules
SL	Concerns of student learning	At least one of the followings is described: (i) Student's ability understand/learn/prove. (ii) Training student's abilities. (iii) Succession/didactical structure (description of an order according to which the proofs shall be executed). (iv) Objective(s) of the lesson (application of theorems and examination focused).
P	Practical concerns	At least one of the followings is described: (i) Time management. (ii) Convenience in teaching/instruction (teacher-centered).
MC	Mathematical content concerns of the proofs	At least one of the followings is described: (i) Structure of proofs, in particular considering the generalization power. (ii) Communicating an image of mathematics in the respective direction.
PN	Concerns of physical natures of the proofs	At least one of the followings is described: (i) Visualising the property/proof. (ii) Pragmatic experiencing the property/proof.
A	Concerns of students' attitudes	At least one of the followings is described: (i) Student's motivation and involvement. (ii) Student's affection towards learning mathematics, in particular, proofs.
	Other/ unrelated	Other reasons not related to the above categories; a serious response but resulting from a misinterpretation of the task; spurious/no reasonable responses; blank or cross out.

Table 13
Rubrics for formal proof formulation

Code	Definition	Coding rules
Visually assisted	Attempt to formulate a proof with figures	A figure is identified.
Deductive proof	A deductive proof is formulated	A logical proof is formed with adequate explanation. No logical/mathematical mistake is found.

Incomplete proof	A proof is formulated but incomplete	A mathematically correct proof is formed without complete explanation; e.g., (i) a figure showing equities of alternate angles; (ii) an attempt that uses the property of exterior angles' sum without justifying it; (iii) a valid proof that considers only some special triangle (e.g. a right triangle).
Unsuccessful	An unsuccessful attempt	At least one of the followings is identified: (i) An attempt is mathematically invalid. (ii) Logical or mathematical mistake is found. (iii) Circular arguments is found.

Appendix 2

Table 14
Rubrics for the scale of knowledge of proofs

Scores	Definition	Coding rules
0–3	Acceptance of preformal proofs in Q1	1 score for accepting each preformal proof presented.
0–2	Ability to formulate a deductive proof in Q2	2 scores for forming a deductive proof; 1 score for forming an incomplete proof; 0 scores for unsuccessful attempt.
0–1	Knowledge of connection	1 score for acknowledging the connection between preformal and formal proofs.
0–6	Total score	The sum of above.

Table 15
Rubrics for the scale of practice of proofs

Scores	Definition	Coding rules
0–4	Preference of proofs in Q3 and Q4	1 score for choosing each proof presented (both preformal and formal proofs).
0–1	Preference of mixed strategies	1 score for demonstrating a teaching approach of using both preformal and formal proofs in class.
0–1	Visual and pragmatic learning experience	1 score for acknowledging that preformal proof can provide students with visual or pragmatic learning opportunity.
0–1	For less capable student	1 score for acknowledging that preformal proof can help less capable students learning by providing alternative learning experience.
0–1	For higher order thinking skills	1 score for acknowledging that preformal can foster higher order thinking, in particular logical thinking.
0–8	Total score	The sum of above.

Table 16
Rubrics for the scale of beliefs about proofs

Scores	Definition	Coding rules
+2	Advocating proofs	An elaborative response which is a positive statement for proof and proving in mathematics teaching and mentions both two of the followings: **Didactical considerations** (i) Didactical consideration - To reproduce in student the problem-solving skills and mental habits of teachers/mathematicians (Larvor, 2010, p. 78) - To encourage independence and self-possession (Larvor, 2010, p. 78) - A divide in the course of mathematical teaching during the standard education (Balacheff, 2010, p. 116) (ii) Consideration about potential issues while dealing with proof and proving in mathematics teaching (e.g. consideration of the connection between the abilities of learners and the proofs being proved), but which does not lead to the conclusion of avoiding proof and proving in mathematic lessons. **Mathematical considerations** (i) Mathematical consideration - To contribute to our understanding of the world around us (Hanna, Jahnke, & Pulte, 2010, p. 3) - Deductively valid arguments leading from true premises to true conclusions (ii) Functional consideration - To produce knowledge: proving theorems through using arguments of deductive logic; e.g. modeling (Hanna & Barbeau, 2010, p. 88) - To create a rupture between mathematics and other disciplines (Balacheff, 2010, p. 115)
+1	Tendency of proofs	A positive response about proof and proving in mathematics teaching, which contains one of the above mentioned. The demarcation to the above scoring is because of less comprehensive and elaborative response that, however, remains positive by justification. If one considers that proof and proving **help** in students' cognitive **developments**, or one shares his/her teaching approaches of (the order of teaching) preformal and formal proofs in Q3 or Q4 (see Remarks), this is coded +1.
0	Neutral	Unjustified response about proof and proving in mathematics teaching. The argumentation does not make clear sense or is incomprehensible. If one means that

		proof and proving is only the chronology/sequence of the curriculum, this is coded 0.
−1	Tendency of rejecting proofs	Respondents tend to reject proof and proving in mathematics teaching. If considerations and reasoning about problems regarding proof and proving in mathematics teaching are a dominant part of the response, it is coded −1 (i.e. non-dominant positive response is allowed).
−2	Rejecting proofs	The meaning of proof and proving in mathematics teaching is judged negatively.

Appendix 3

Questionnaire for respondent's demography

Demographics (mark a "√" for your choices)

My gender is ☐ Male ☐ Female.

My e-mail is (optional):

1. In which year of study (semester) are you?
   ☐ year 1 ☐ year 2 ☐ year 3 ☐ year 4 ☐ PGDE
   ☐ others
2. What is your minor subject (second subject) beside mathematics?
   ☐ Sciences ☐ Languages ☐ Social Sciences ☐ PE ☐ Music
   ☐ Art ☐ Others
3. What have you already learned in mathematics/mathematics education in your undergraduate and/or PGDE/PGCD studies?
   Mathematics:
   ☐ Number Theory
   ☐ Essential Mathematics Concept/discrete mathematics/finite mathematics
   ☐ Geometry/Geometry and Measurement
   ☐ Linear Algebra/abstract algebra
   ☐ Probability
   ☐ Recreational Mathematics
   ☐ Mathematical Analysis/introductory analysis/real analysis
   ☐ Others
   Mathematics education:
   ☐ Curriculum in mathematics
   ☐ Teaching and learning in mathematics
   ☐ Assessments in mathematics
   ☐ Mathematics problem solving/problem solving and its enhancement
   ☐ History of mathematics and its relationship with the teaching of mathematics
   ☐ Use of ICT in mathematics teaching
   ☐ Psychology of children/adolescence
   ☐ Learner's differences (talented and those with learning difficulties) in mathematics
   ☐ Mathematics activities, games and explorations
   ☐ Mathematics classroom Management
   ☐ Pedagogical Content Knowledge in mathematics teaching
   ☐ Others
4. Do you already have any practical teaching experience (school practice/other related

experience)?
☐ Yes ☐ No
If yes, what kind of?
☐ As a student teacher in a school practice (e.g. Field experience)
☐ As a subject instructor in a private institute (e.g. Tutor in tutorial school)
☐ As a teaching assistant in a school
☐ As a lab demonstrator or tutor in a university
☐ Others

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[1] French textbook: Hatier, (1998), *Le nouveau Pythagore class 4ème*r, p. 165.
[2] Hong Kong textbooks: Chung Tai Educational Press. (2015), *Effective Learning Mathematics* (volume 1A), p. 4.17; Educational Publishing House Limited. (2013). *Mathematics in Focus* (volume 1A), p. 4.23; Hong Kong Educational Publishing Co. (2013). *New Progress in Junior Mathematics* (volume 1B), p. 5.21; Pearson Educational Asia Limited. (2007). *Mathematics in Action* (volume 1A), p. 238; Oxford University Press (China) Ltd. (2009). *NEW CENTURY Mathematics* (volume 1A), p. 273.
[3] Proposition 1.13 states the following: 'If a straight-line stood on a(nother) straight-line makes angles, it will certainly either make two right-angles, or (angles whose sum is) equal to two right-angles' (Fitzpatrick, 2007, p. 18). Proposition 1.29 states the following: 'A straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles' (ibid., p. 32). Proposition 1.31 allows one 'to draw a straight-line parallel to a given straight-line, through a given point' (ibid., p. 33–34).

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