Learning from Teaching Teachers: A Lesson Experiment in Area and Volume with Prospective Teachers

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Learning from the practice of teaching teachers is a promising approach for mathematics teacher educators to attain the extensive knowledge and abilities needed for their roles. As mathematics teacher educators ourselves striving to develop across our professional lifespans, we used one such approach, known as a lesson experiment, to investigate how our instruction affected prospective elementary teachers' conceptual understandings of area and volume. Classroom evidence included audio recordings of ten prospective teachers' work on lesson activities as well as written work on three post-assessments. Our analysis for the lesson experiment consisted of two phases, first determining to what extent the learning goals were achieved and then evaluating our hypotheses and instruction for how they supported or hindered the learning goals. Findings revealed the prospective teachers enhanced their understandings of area and volume, yet some unexpected complexities in their thinking arose. The lesson experiment led to instructional recommendations for improving the lesson and enhanced our knowledge as mathematics teacher educators. Implications include the value of lesson experiments as a reflective process to help mathematics teacher educators develop over time as well as practical advice for helping prospective teachers enhance their understandings of area and volume.

Keywords · prospective teachers · lesson experiment · area and volume · learning from teaching · mathematics content course

The mathematical preparation of teachers is a significant international topic (Even & Ball, 2009). There is general agreement that teachers need a deep knowledge of mathematics for teaching (e.g., Delaney, Ball, Hill, Schilling, & Zopf, 2008) and that mathematics courses in teacher preparation programs serve as one site to provide such learning opportunities (National Council of Teachers of Mathematics [NCTM], 2007). However, a challenging question is how do *mathematics teacher educators* learn to offer instruction that helps teachers develop such mathematical understandings? (Anthony, Cooke, & Muir, 2016; Zaslavsky, 2008). The knowledge demands and abilities required of mathematics teacher educators are extensive and multi-faceted, as they need to know much of what is known by mathematics teachers and more (Beswick & Chapman, 2015; Jaworski, 2008). They need a thorough knowledge of mathematics, of mathematics pedagogy for grade K-12 students as well as prospective and practicing teachers, of curriculum and assessment practices for primary through tertiary levels, of professional and research literature related to the teaching and learning of mathematics, and of research methodologies for examining situations of mathematics learning and teaching. Mathematics teacher educators do not attain such understandings in a short period of time, nor do they tend to complete formal preparation programs (Cochran-Smith, 2003). Rather, most mathematics teacher educators are ‘self-made’, transitioning from mathematics teachers or researchers of mathematics learning and teaching, through a process extending across their professional lifespans (Cochran-Smith, 2003; Zaslavsky, 2008; Zaslavsky, Chapman, & Leikin, 2003). One such means is to ‘learn from doing’ or for mathematics teacher educators to learn from their practice of teaching teachers and thereby enhance their instruction over time (Bragg, 2015; Chapman, 2008; Chick & Beswick, 2013; NCTM, 2007; Richert, 1995; Tzur, 2001; Zaslavsky, 2008).

Learning from one’s teaching practice requires more than just experience. To take advantage of experience, mathematics teacher educators need to be reflective practitioners (Schön, 1987). Chapman (2008) conducted a review spanning ten years of studies in which mathematics teacher educators researched their own instructional practices with prospective teachers. She found in order for mathematics teacher educators to learn from investigations of their practice, the research needs to a.) focus on an aspect that is part of the teacher educator’s regular teaching, b.) expose the prospective teachers’ thinking during the process, c.) include self-reflection by the teacher educator before, during, and after the instruction, and d.) examine critically the prospective teachers’ learning and the role of the instruction in promoting it. Such research enables teacher educators to know something significant about their practice that they did not know before. “It can also inform others of what mathematics teacher educators can learn or have learned, and how they learn, from conducting these studies” (Chapman, p. 115).

As mathematics teacher educators developing across our professional lifespans, we often seek ways to learn from our teaching practice, asking what we would like to know more about in terms of teaching mathematics to prospective teachers (PSTs). During one such conversation, we decided that we wanted to know more about how our instruction impacts PSTs’ understandings of area and volume. In addition, we decided to do so using a *lesson experiment* (Hiebert, Morris, & Glass, 2003), a process that meets the conditions reported in Chapman (2008). The purpose of reporting our efforts here is two-fold. First, we present our findings with regard to how our instruction affected PSTs’ mathematical understandings of area and volume. Second, we share our experiences with and lessons learned from undertaking a lesson experiment, with the intent to inform other mathematics teacher educators of ‘how we learned’ (Chapman, 2008, p. 115).

Lesson Experiments

A lesson experiment is an “intentional, rigorous, and systematic process of learning to teach through studying one’s own practice” (Hiebert, Morris, Glass, 2003; p. 201). Instructors engage in cycles of testing hypotheses about cause and effect relationships between teaching and learning in order to address, “What did students learn, and how and why did instruction influence such learning?” (Hiebert, Morris, Berk, & Jansen, 2007, p. 48). The experiment is applied to an instructional episode such as a specific task, a full lesson, or a sequence of lessons. It is composed of four steps akin to teacher as researcher, reflective practice, and disciplined inquiry. The first step consists of explicating the learning goals and specifying hypotheses of how instruction may support those goals, both of which then inform the planning of the lesson. The second step is assessing to what extent students achieve the learning goals by gathering during the lesson and analysing afterwards evidence of students’ thinking from videos, transcripts, or written work. The third step consists of evaluating the hypotheses for why the lesson did or did not achieve the learning goals. The fourth step entails using the hypotheses to revise the current or future lessons. These steps shift an instructor’s focus from teaching in the moment to including preparation and reflection outside the classroom. Hiebert et al. find promise in the approach as a.) its steps are taken from the (implicit) practice of classroom teachers, b.) it melds the activity of teaching and research, c.) it offers instructors the opportunity to gain knowledge for improving their own teaching over time, and d.) it works with all types of learning goals. While Hiebert and his colleagues foremost recommend lesson experiments to help teachers learn from teaching, they also recommend the approach for university teacher educators (e.g., Phelps & Spitzer, 2012).

Our Lesson Experiment Process

We are two mathematics teacher educators at a mid-sized research university in a rocky mountain region of the United States. One of our primary roles is to offer mathematics courses designed for elementary education majors. At the time of the lesson experiment, the first author was as an assistant professor in the mathematics department. Her background included a doctoral degree with a mathematics education emphasis in Curriculum and Instruction, a master's degree in mathematics with an outside specialisation in education, and a bachelor's degree in mathematics. She had been offering mathematics courses for elementary PSTs for eight years, including serving as the course coordinator as well as instructing the geometry and measurement course three times, the number and operations course ten times, and the algebra, data analysis, and probability course five times. The second author was a second-year doctoral candidate in the mathematics education program. She had a master's degree in mathematics, and it was her second time teaching the geometry and measurement course.

The semester before the lesson experiment, we engaged in various teaching discussions as part of our course coordination meetings. Amidst those conversations, we realised we wanted to learn more about how our instruction affected PSTs' understandings of area and volume. Using a lesson experiment to do so appealed to us for many reasons. First, as a lesson experiment may be applied to a single lesson rather than having to span a full unit or course, it felt manageable and fit well with our other professional demands that semester. Second, a lesson experiment appeared to be more feasible for two individuals than other models of teacher research, such as lesson study (Lewis, 2002), which requires a large team of teachers working together. Finally, and most importantly, we felt a lesson experiment meets the conditions listed by Chapman (2008). Specifically, lesson experiments emphasise collecting artefacts of students' learning during pivotal points of the lesson in order that the instructor may reflect before, during, and after about how the instruction affected students' understandings. For these reasons, we used a lesson experiment to investigate the impact of instruction on our PSTs' understandings during one lesson on area and volume.

Our lesson experiment setting was a 3-credit geometry and measurement course, taken as the third mathematics content course for elementary education majors. The second author was the instructor, while both authors served as researchers. Of the 19 PSTs in the course, 63% were sophomores, 26% juniors, and 11% seniors or post-baccalaureates with backgrounds ranging from high school algebra to calculus. The lesson experiment occurred within the first unit, Measurement Processes and Systems. Our planning for this unit was heavily influenced by Inskeep (1976) and Van de Walle (2007), which describe a progression for elementary students learning to measure with understanding (see Figure 1). The progression begins with students being able to perceive an attribute, fostered through direct comparisons with no numerical referents. The next step incorporates the need for and use of a unit, generated by providing scenarios in which direct comparison is no longer feasible. The goal is for students to understand what might serve as a unit as well as to produce a number called a measure by matching repetitions of the unit to the attribute. Such activities may begin with non-standard units and then progress to standard units as students are ready. The final step involves students working within measurement systems that structure the standard units and include standard measuring tools.

Plan for Measurement Instruction (PMI)

Goal 1: Students will understand the attribute to be measured.

- **Type of activity:** Have students make comparisons of the attribute with different objects. For example: Which is longer/shorter? heavier/lighter? holds more/less? Use direct comparisons whenever possible.

Goal 2: Students recognise the need for and use units to produce a measure.

- **Type of activity**: Present students with objects to compare and measure for which direct comparison is no longer feasible to generate the need for and understanding of a unit. Have students measure physical models of the attribute using nonstandard and then standard units. Include activities where students have multiple copies of the unit available and where students have only one copy of the unit available.
- **Comment:** The teacher can help students understand the need for a common unit by asking them to measure a single object with different sized units.

Goal 3: Students will use common measuring tools, measurement systems, and formulas with understanding and flexibility.

- **Type of activity:** Have students make their own measuring instruments with informal units and then compare how those are measuring in the same way that standard instruments do; allow students to measure physical objects with the standard units; plan activities for students to develop familiarity with standard units.
- **Comment:** Introduce measurement formulas and unit conversions only after students fully understand how to directly measure the attribute.

Figure 1. Plan for Measurement Instruction.

We utilised the PMI in our Measurement Processes and Systems unit in two ways. First, we wanted our PSTs to use it when addressing measurement with their future students. Upon presenting similar frameworks to primary teachers, Clarke, Cheeseman, McDonough, and Clarke (2003) found that teachers reported greater use of open-ended questions, more connections among mathematical ideas and real-life mathematics, less emphasis on formulas and procedures, and enhanced interpretations of students' thinking on measurement tasks. Unfortunately, teachers often lack any explicit knowledge of such measurement learning frameworks (O'Keefe & Bobis, 2008). We therefore included the PMI as one of our learning goals for the PSTs (see Table 1 below). Second, we followed this sequence with the PSTs for each of the attributes addressed in the unit. We felt doing so would enhance the PSTs' mathematical understandings as well as allow them to experience the progression as learners themselves.

Step One: Learning Goals, Hypotheses, and Lesson Planning

To develop the Measurement Processes and Systems unit, we consulted resources providing learning goals for PSTs (Conference Board of the Mathematical Sciences (CBMS), 2012; Sowder, Sowder, & Nickerson, 2014) and for elementary and middle school students (NCTM, 2000; Van de Walle, 2007). We articulated enduring understandings (Wiggins & McTighe, 2005) that we intended to address across the entire unit as well as knowledge and skills objectives specific to each attribute (length, area, volume, angle, weight, mass, time, and money). Table 1 provides the enduring understandings for the entire unit, which we addressed at various times in various lessons across the unit. The lesson for our lesson experiment specifically addressed enduring understandings one, two, three, and six with regard to area and volume. Table 2 presents the knowledge and skills objectives specific to our lesson experiment for area and volume (we addressed additional area and volume objectives in future lessons). For the remainder of the paper, we use the term *learning goals* to refer to the enduring understandings and knowledge and skills objectives specific to our lesson experiment.

Table 1
Enduring Understandings across Unit 1: Measurement Processes and Systems

Topic	Enduring Understandings
1. Process of measuring	EU1a: To measure an attribute of one or more objects that cannot be compared directly, some means of comparison is needed, often a unit of the same attribute. EU1b: Measuring is the act of counting repetitions of a unit until the number of repetitions matches the attribute being measured. It results in the quantification of an attribute in relation to a constant unit. EU1c: The size of a unit and the measure of an attribute are inversely related. EU1d: Additivity of Measures: An attribute may be measured by being separated into a number of parts that are then each measured and summed.
2. Standard units	EU2a: Standard units are necessary for communicating consistently about an attribute. EU2b: Using a standard unit ensures that the same attribute will result in one measure.
3. Measurement systems	EU3: A measurement system allows one to measure attributes with a unit appropriate for the context, including the ability to measure an attribute with different magnitudes.
4. Precision	EU4a: Measurements are approximate, and the selection of a unit affects precision. EU4b: Context often determines the needed degree of accuracy.
5. Measurement formulas	EU5a: A measurement formula allows one to measure an attribute without counting repetitions of a unit. As such, they are often more efficient. EU5b: Measurement formulas are based on the properties of the object.
6. Supporting elementary students in learning measurement (PMI)	EU6: In learning to measure and to understand measurement systems, students are supported by a progression of a.) learning to perceive an attribute, b.) comparing objects with the same attribute, c.) measuring with a unit (nonstandard and then standard), and d.) working within a standard measurement system.

Table 2
Knowledge and Skills Objectives for Area and Volume Specific to the Lesson Experiment

Knowledge (K)	Skills (S)
K1a: Area is the two-dimensional space inside a region. K1b: Area is measured by the number of square units that cover the region. K2a: Volume is the space inside a three-dimensional solid. K2b: Volume is measured by the number of cubic units (solid or liquid) that fill the space.	S1: Compare, order, and measure area and volume using nonstandard and standard units (including selecting an appropriate unit or tool). S2: Develop familiarity with standard units for area and volume (metric and US Customary).

Three hypotheses, about how instruction might support the PSTs in attaining the learning goals, guided our planning of the lesson:

1. As PSTs often bring prior understandings of area and volume, it is not necessary that all PSTs complete all area and all volume activities. Rather, completing an area station or a volume station (as determined by performance on a pre-assessment) followed by a presentation on the other attribute will address each PST's remaining needs with respect to area and volume while conserving limited classroom time.
2. If the PSTs complete area and volume activities in sequences aligned with the PMI, they will enhance their understandings of how to measure area and volume, e.g., attain the learning goals presented above.
3. The PSTs will better understand the pedagogical implications of the PMI if they experience the progression as learners themselves, have illustrative activities to refer to when being introduced to the plan, and are asked to interpret additional measuring activities with respect to the plan.

We began the Measurement Processes and Systems Unit with a pre-assessment (Hypothesis 1). Our lesson experiment occurred after taking the PSTs through the PMI sequence for angle and length (Hypothesis 3). We began the lesson with the instructor describing the PMI, referring to relevant angle and length activities. Next, the PSTs completed a station in either area or volume (Figure 2). We designed the stations to align with the PMI (Hypothesis 2) and to address our learning goals (see Table 3). During the stations, the instructor observed the groups, offering clarification and guiding questions as needed. After the stations, each volume group presented their findings to an area group and vice-versa (Hypothesis 1), explaining how to measure their attribute, the metric units they used, and how their activities related to the PMI (Hypothesis 3).

Area Station

1. Which is the Largest? Cut out the triangle, square, and circle. Determine some way (without using a ruler) to verify which of the shapes has the largest area. Explain.
2. Verbal Communication: Inside the bag is an item to be measured, along with a measuring tool. Use the tool to measure the area of your object. Without revealing your objects, compare the number you obtained with the other area group. Based on your numbers, which group's object has a larger area? Now, compare your actual objects. What can you say about the actual area of your objects? Is this different from what your measurements told you? Why? (Both groups measured the same rectangle with either 7.5-cm or 5-cm square sticky notes.)
3. Area Treasure Hunt: We are going to use the faces of the base-ten blocks to measure area. Imagine spreading ink on the face of a base-ten block and 'stamping' out the associated surface area when you approximate. First, use the base-ten blocks to measure the surface area of the following objects: a piece of paper, the top of a CD case, the front of a standard door, the bottom of a size seven shoe, a table top, and the top of a laptop. Second, for each of the following, find or think of the (surface) area of an object which fits the given area measurement and then explain your reasoning: ten square centimetres, one-hundred square centimetres, one square decimetre, three square decimetres, and one square metre. Do you think the base-ten blocks are a better or worse measurement tool than the unit in Step 2? Why? How comfortable are you measuring area within the metric system?
4. Guess the Unit: Try to guess what unit was used in each of the following measures:
   a. The surface area of an average human fingernail is about 50 _______.
   b. The area of a football field is about 50 _______.
   c. The average surface area of the top of a coffee table is about 85-90 _______.
   d. The surface area of a human palm is about 130 _______.

Volume Station

1. Popcorn Volume: Create three cylinders from the pieces of paper. For the first cylinder, tape the long sides of the paper together. For the second cylinder, tape the short sides together. For the third cylinder, cut the paper in half and tape the two pieces together for a short cylinder. Use the popcorn to determine which of the three cylinders has the greatest volume. Explain.
2. Verbal Communication: Same as Area Step 2 but with volume units. (Both groups measured their second cylinder with either standard or miniature peanut butter cups.)
3. Base-Ten Blocks: Find a way to use the base-ten blocks to measure the volume of your second cylinder. What measurement did you determine? What units are you using? Do you think the base-ten blocks are a better or worse measurement tool than what you used in the previous steps? Why? Is measuring the volume of the cylinder with the base-ten blocks "good enough"? Why or why not? What might be a better way to measure volume?
4. Water, Water Everywhere: Use the litre of water and hollow base-ten cube to determine how we can convert from a cubic unit (a base-ten block) to a liquid measure. How much water fits into a hollow base-ten cube? A hollow base-ten unit?

Figure 2. Volume and Area Stations.

To determine the station each PST completed, we relied upon the pre-assessment, which asked the PSTs to explain area and volume and to match various objects with area and volume measures and units. PSTs with more significant misconceptions in area (e.g., equating area with $length \times width$, descriptions such as "how much space is taken up") we placed in the area groups, while those with more significant misconceptions in volume (e.g., equating volume with $length \times width \times height$, confusing volume and weight) we placed in the volume groups. If PSTs held misconceptions in both areas, we placed them in the area group as area may serve as a precursor to volume (Lehrer, Jaslow, & Curtis, 2003). Of the ten PSTs for whom we analysed lesson experiment data, only four described area as the space a two-dimensional shape covers, six included descriptions of area as $length \times width$, four confused area and volume, and none mentioned that area is measured by counting square units. Five PSTs described volume as "how much a three-dimensional object can hold", three alluded to volume but included area language, five included $length \times width \times height$, and none explained volume is measured in cubic units.

Table 3
PMI Goals and Intended Learning Goals for Each Step in the Stations

Step	PMI	Learning Goal(s)
1	Goal 1 – comparison activities	EU1d, K1a or K2a, S1 (compare and order)
2	Goal 2 – nonstandard units	EU1a-c, EU2, K1ab or K2ab, S1 (measure)
3	Goal 2 – standard units	EU1b, K1ab or K2ab, S1 (measure), S2
4	Goal 3 – measurement system	EU3, K1ab or K2ab, S2

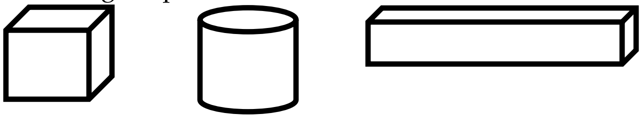
Step Two: Gathering and Analysing Evidence of PSTs' Thinking

We collected data from ten PSTs; five completed the area station as one group, and the other five completed the volume station as one group. We gathered audio-recordings and the PSTs' written group work on the stations. We also gathered the ten PSTs' individually written work on three post-assessments: a follow-up worksheet, a homework assignment, and an in-class test (Figures 3, 4, and 5 and Table 4). The PSTs completed the follow-up worksheet after the area and volume presentations. The PSTs completed the homework and test near the end of the unit because they included items on area and volume as well as other attributes addressed in the unit. While we acknowledge the homework and exam may include learning by the PSTs' after the lesson experiment, we decided to include them in our analysis for two reasons. They provided information about the individual PSTs' understandings of area and volume (since the stations only provided a group perspective), and they revealed remaining misconceptions in the PSTs' understandings of area and volume that we felt were relevant for revising the lesson. Our analysis of the station data and the post-assessments consisted of two phases, first determining the extent to which the learning goals were achieved (Step 2 of a lesson experiment) and then evaluating our hypotheses and instruction for how they supported or hindered the learning goals (Steps 3 and 4).

Area and Volume Follow-Up

1. Which figure has the larger area? Why?

2. How would you prove which of the shapes is larger without using a measuring tool such as a ruler?
3. Which of the following shapes would be the best unit to use to measure volume? Why?



4. Explain why the volume of a shoebox is "*length x width x height*".

Figure 3. Area and Volume Follow-Up

Homework (exclusive to items relevant to area and volume)

1. Describe area and then volume. Be extremely detailed!
2. For the following groups of objects, determine the attribute(s) being measured, give appropriate English AND metric units for measuring the attribute, and determine an ordering from smallest to greatest or explain why this is not possible.

Group One
The amount of wallpaper for a bedroom wall
The amount of sod for a football field
The amount of plastic wrap over a cake pan

Group Two
The amount of yarn used to make a scarf
The amount of hair cut off in a typical haircut
The amount of thread in a friendship bracelet

Group Three
The amount of matter in a piece of bread
The amount of water you drink in a day
The amount of food you eat in a day

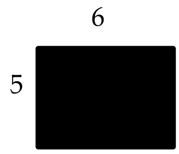
Group Four
A handful of rabbit fur
A thimbleful of lead
A blown-up balloon

Group Five
A tank of gas
A dose of cough medicine
A bottle of juice

Group Six
Pack for a week-long trip
Brush your teeth
Cook a turkey

Group Seven
How heavy a Christmas present is
How big a Christmas present is
How long it takes to open a present

3. What is the area of the rectangle? How do you know?



4. What is the area of the rectangle? How do you know?

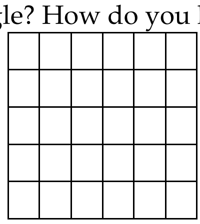
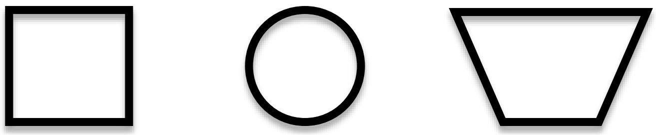


Figure 4. Homework Assignment

Unit Test (exclusive to items relevant to area and volume)

1. Describe the pros and cons of using each of the following as a unit for measuring area.



2. Order the following steps students commonly go through when learning measurement:
   - _____ Working within a standard measurement system
   - _____ Comparing objects with the same attribute
   - _____ Measuring with a non-standard unit
   - _____ Learning to perceive an attribute
   - _____ Measuring with a standard unit
3. Which step of the measurement learning progression does the following activity address? Explain.

   Elementary students are asked to match the correct unit of length (centimetre, metre, kilometre, etc.) to various objects they might wish to measure, such as the distance from their house to the school, their height, etc.
4. Identify the attribute being measured (choosing from length, area, volume, angle, mass, weight, or time) in each example.
   a. The circumference of a circle
   b. The wallpaper needed to wallpaper a bedroom
   c. The amount of thread used to sew on a button
   d. The length of a movie
   e. The amount of matter that makes up a dog
   f. The amount of vaccine given to an infant
5. Imagine a box that is ten centimetres wide, eight centimetres long and seven centimetres tall. We say the box has a volume of $10 \times 8 \times 7 = 560$. What are the units on this number?
6. Match the following measurements with the object they most likely measured:

	1. Volume of a base-ten cube
a) _______ one gram	2. Length of a football field
	3. Mass of a standard paper clip
b) _______ two square-metres	4. Amount of wallpaper to cover a classroom wall
	5. Surface area of a standard door
c) _______ one litre	6. Mass of a bottle of water
	7. Distance from 3rd street to 20th street
d) _______ two kilometres	8. Liquid in an extra-large jug of juice

7. Say the volume of an object is measured to be 1200. What does this number mean? (Don't just say it is the volume …)

Figure 5. Unit Test

Table 4
Learning Goals Matched with Post-Assessment Items

Goal	Follow-Up	Homework	Test
EU1	1, 2	2, 3, 4	6, 7
EU6			2, 3
K1	1, 2	1a, 2, 3, 4	1, 4, 6
K2	3, 4	1b, 2	4, 5, 6, 7
S1	1, 2	2, 3, 4	5
S2		2	5, 6

Data Analysis Phase 1: Achievement of Learning Goals

To determine the degree to which the learning goals were achieved, we examined the PSTs' thinking on the area and volume stations and then on the post-assessments. We began by listening through the audio-recordings to generate a summary of each group's work on the stations. The summaries included explanations of the PSTs' mathematical thinking, questions or comments that caused the PSTs to review their previous work, and interactions between the groups and the instructor. At times, we loosely transcribed comments from the PSTs or the instructor when their words held significant implications for the PSTs' thinking or mathematical conclusions. We listened through each audio-recording at least four times, adjusting the written summary as needed. Figure 6 provides an example of our summary for the volume group's work on Step 3. We then moved to more interpretative analyses, making memos about the PSTs' work against normative mathematical answers as well as looking for patterns in how the PSTs completed similar parts. For example, we looked for patterns in how the area group determined the missing units in Step 4. We noted that throughout, "there seems to be quite an emphasis on linear interpretations, e.g., seeing the numerical measurements as length dimensions rather than area measurements." Finally, we created a table summarising the PSTs' understandings after the stations relative to the learning goals. Each row addressed one of the learning goals and the associated column described the group's understandings relative to that goal. We synthesised our interpretive memos and the table into written descriptions of the area and volume stations, including details about the PSTs' mathematical thinking, their general approaches, assessments of their work, and their understandings relative to the learning goals.

Time: 28:20
PSTs read the directions for Step 3. They get base ten blocks to measure the volume of the second cylinder.
PST: Should we use the same measurement as last time?
They find height (13) and diametre (7), then radius (3.5). Then they use the formula to compute: $pi \times 3.5 \times 13 = 500.045$ cm³.
PST A: cubic centimetre
Another PST: yeah, that's right because these are cm cubed
PSTs together: cm cubed
PSTs talk about how the base ten blocks are in centimetres, so centimetres cubed
PST A says he would be surprised if someone told him the cylinder holds 500 units.
Another PST: Is that right?
PST A: Not necessarily
Another PST says she believes they would fit and begins counting. As a group they begin putting in base-ten longs – get 80 in, which take up less than 1/5 of the space and PSTs look at how much space is left.
PST A brings up cramming little cubes into a round thing and all the space that will be left over.
PSTs work on putting blocks in the cylinder. Other PSTs double check multiplication from previous step. PSTs talk about how 250 units should fill ½ of the cylinder. They ask, does that look like half? Some PSTs say it looks like more. PSTs keep putting blocks in the cylinder. They note the several spaces between the blocks. They realise their total gets close to 500. (They get 430 base-ten units in the cylinder.)
PSTs discuss that next time they're going to melt these down and pour them in.
PSTs work on answering the questions from Step 3 – see their written work.
Time: 36.00

Figure 6. Example of Written Summary for Volume Group Step 3

For the post-assessments, we began with a coding process and then moved into a conclusion-drawing phase. For the coding, we read through the PST's responses on each post-assessment item, using codes to denote similar ways of thinking, procedures, and explanations as well as misconceptions, shortcomings, and issues. For example, on Follow-Up #2 we had the code *Square Units* for responses comparing the number of square units that fit inside each shape and the code *Fail to Attend to Both Dimensions* for responses that attended to only one of the linear factors. We also coded the learning goals addressed by each PST's response. Throughout, we adapted our codes as needed, returning to previous codes to make sure they were consistent. For the conclusion-drawing phase, we pulled together for each learning goal all of the post-assessment items that included a relevant response from the PSTs. Then, for each way of thinking or misconception associated with a particular learning goal, we tallied frequency data about how many and whom of the PSTs demonstrated the associated way of thinking or misconception. Using such frequency information, we created descriptions of the PSTs' understandings on the post-assessments with respect to area, volume, and the enduring understandings.

Data Analysis Phase II: Evaluation of Instruction

To evaluate our hypotheses for supporting the PSTs in attaining the learning goals, we returned to our summaries of the stations and the post-assessments. First, we made memos about how the steps in the stations affected the PSTs' thinking as well as ways to enhance the stations. For example, we noted that because the area station asked the PSTs to find the area of a sheet of paper, they used the paper to measure the other larger objects. This inadvertently led to the PSTs encountering obstacles with the formula $length \times width$ and with using a non-square area unit. Second, we turned to misconceptions, shortcomings, and issues identified in the post-assessments. We looked back at the stations to find affordances and hindrances for such issues. For example, Test #1 revealed that the PSTs did not fully appreciate the advantages of using a square for an area unit. We therefore returned to the area station summary and recognised that Step 2 just presented PSTs with a square area unit rather than asking them to consider possible shapes for an area unit. These issues pointed to instructional improvements, some of which we generated on our own and some of which we adapted from returning to the literature on PSTs' and children's understandings of area and volume. We gathered our memos on how the instruction affected the learning goals and our associated instructional adjustments, organised them by their relevance to each of our three hypotheses, and prepared an explanation of whether and how each hypothesis supported the learning goals.

Results: Degree to Which Learning Goals Achieved

PSTs' Work on the Area Station

On area Step 1, the PSTs were initially swayed by the side lengths and diametre, incorrectly deciding the triangle was larger because its sides extended over the edge of the circle. To which the instructor asked, "Can I do anything with these to figure out which one fits inside?" Upon deciding to cut off the extended parts and see how they fit over the other shape, the PSTs correctly ordered the shapes by area, understanding that area is the amount of space inside a two-dimensional region and that the area of a region remains the same upon being cut into smaller pieces. For Step 2, the PSTs covered the rectangle with whole and then cut up pieces of 5-cm square sticky notes, determining an area measurement of 7 ½ sticky notes. Another area group shared their measurement of $3\frac{1}{9}$ sticky notes. Both groups believed the measurement of 7 ½ was for a larger shape, but upon revealing their units and shapes found that the other group used larger sticky notes (7.5-cm square). The PSTs gained an appreciation of standard units for communication, understood that different units lead to different measurements, and realised smaller measurements correlate with larger units.

On area Step 3, the PSTs' approaches varied by the size of the object to be measured. When the PSTs measured smaller regions, they covered the regions with area units and thereby found reasonable area measures. For example, they counted the base-ten blocks that covered a piece of paper and determined its area to be "588" (with no mention of the units cm^2; a possible indication of lack of understanding of the square unit – as described more below). Whenever the PSTs measured larger regions, they did so using the sheet of paper (8.5 inches by 11 inches) to measure the length and width and then incorrectly added the length and width dimensions or failed to recognise the complexities when using a rectangular area unit (how the linear units constitute the area unit). For example, they estimated the front of a standard door to be 7 feet by 3.5 feet or approximately 7 sheets high and 3.5 sheets wide. Thus, the PSTs used the longer side of the paper to measure height and width and unknowingly created an 11-inch square area unit, not the sheet of paper. The PSTs then incorrectly added the linear dimensions, taking $7 + 3.5 = 10.5 \approx 11$ and 11×588, arriving at too small of a measurement, "6,468" cm^2.

On the second half of Step 3, the PSTs often had difficulty visualising the given measurements in terms of square units and sought the instructor's assistance. First, the instructor demonstrated that the face of a base-ten unit was one square centimetre because each side had a length of one centimetre. A little later, due to the PSTs' uncertainty of 1 dm^2, the instructor began with their knowledge that 10 cm = 1 dm and asked if we want to measure area, what do we need to do? The PSTs suggested "square it" so the instructor wrote $(10 \text{ cm})^2 = (1 \text{ dm})^2$, which implies $100 \text{ cm}^2 = 1 \text{ dm}^2$ or a base-ten flat. After such demonstrations, the PSTs proceeded to select reasonable objects for the given area measures, except for the last measure of 1 m^2. The PSTs converted 1 m^2 to 10,000 cm^2 and decided the front of the large door in the classroom may be close to 10,000 cm^2. Without measuring, they discussed that the door was approximately 10 ft. x 3 ft. or about 3.5 m by 1 m, so 3.5 m^2. The PSTs did not compare 1 m^2 with this estimate, nor did they visualise 1 m^2 as a 1 m by 1 m square, instead visualising it only as 10,000 cm^2. At the end of Step 3, the PSTs wrote, "base-ten units are better because they are a standard unit of measure".

Throughout Step 4, the PSTs viewed the quantities as length measurements rather than as area measurements and determined an appropriate unit by selecting the unit between too large and too small of a unit. For example, to determine the unit for the area of a football field, the PSTs perceived 50 as a length measurement of the field rather than the number of square units covering the field. They ruled out kilometres as being too big (relating kilometres to miles) and metres as being too small, so they selected decametres. Throughout, the PSTs did not consider that 50 might be the product of two linear factors, e.g., a 5 x 10 rectangle. Nor did they appear to have a sense of the size of the area unit, for example that a square decametre is 1 dam by 1 dam. When writing their units, the PSTs failed to square them. The PSTs did understand the order of metric units and that a measurement system includes different sized units for measuring attributes of different magnitudes.

PSTs' Work on the Volume Station

For Step 1, the PSTs expected the volumes to be equivalent since they were created from the same size sheet of paper. However, they concluded, "C had the biggest volume. When we filled A to the top, the same amount of popcorn did not fill B and the same amount again only filled C to about half." Upon questioning the instructor about the different volumes, she engaged them in a discussion of the cylinder volume formula ($V = \pi r^2 h$) and the greater impact of changes in the radius versus changes in the height.

On Step 2, the PSTs measured the volume of the second cylinder in three ways using miniature peanut butter cups (cylindrical with diametre of 1.25 inches). First, they used the diametre of the cups to measure the height and the radius of the cylinder. As such, they unknowingly utilised a 1.25-inch cube as their volume unit, failing to realise how the linear units constituted the volume unit. The PSTs then used the cylinder volume formula, $\pi \times 1.25^2 \times 8 = 39.27$ or "39 peanut butter cups cubed" (redundant as the cups were already a volume unit). Second, the PSTs found that 28 peanut butter cups fit in the cylinder. Finally, they formed a layer of cups on the bottom of the cylinder and stacked them, resulting in 32 peanut butter cups. For the filling and stacking methods, they discussed that gaps remained between the peanut butter cups and therefore felt more assured of their formula measurement. Upon learning that the other volume group had a measurement of 3.06, a PST remarked, "They measured with elephants", likely revealing an a priori understanding of the inverse relationship. After confirming that both groups measured the same cylinder, the PSTs realised that the different measurements resulted from different sized units and held an appreciation for standard units.

To measure cylinder two for Step 3, the PSTs aligned the base-ten units with the height and radius and then computed $pi \times 3.5^2 \times 13 \approx 500$ cm^3. One PST commented he would be surprised if 500 base-ten units fit in the cylinder, so the PSTs proceeded to do so, resulting in "430". The PSTs again recognised that there were gaps between the base-ten blocks and therefore decided that 500 cm^3 was a reasonable volume measure. They explained that while base-ten blocks are a better unit than peanut butter cups, water displacement might be the best unit for measuring volume since it eliminates gaps.

For Step 4, the instructor asked the PSTs how many base-ten units fit in a base-ten cube, to which they replied "1000". The PSTs then filled the hollow base-ten cube with the water from the 1 L bottle. The instructor clarified that 1000 cm^3 are therefore the same as 1 L. After a brief reminder that there are 1000 mL in a L, she asked, "There are 1000 mL in a litre and 1000 cubic centimetres in this cube, so how many millilitres in this [holds up a cubic centimetre]?" The PSTs answered "one" and wrote, "1 litre fits into a hollowed-out base-ten cube, so 1 litre is approximately 1000 cm^3. A hollowed-out base-ten unit would equal $1/1000$ cm^3 [should be dm^3] or 1 mL." The PSTs used the numerical scale of $1/1000$ to realise that 1 cm^3 is equal to 1 mL. It was not possible to tell to what degree they visualised 1000 cm^3 in the base-ten cube, 1000 mL in a litre, or 1 mL in a hollow base-ten unit.

Presentations

For the presentations, the area group described their work above and explained that circles do not work well as an area unit because gaps remain. They then explained that their activities related to the PMI in "being able to estimate and understand what the measurement is." The volume group also explained their work above; however, they did not elaborate on how the formula explains why the cylinders had different volumes. They also explained how their steps aligned with the PMI, "Since seeing how much stuff would fit in each cylinder, they [elementary students] would understand that the volume would be what is the space inside." Neither group commented on how any of the steps related to the specific goals of the PMI.

PSTs' Conceptions of Area as Revealed on the Post-Assessments

Across the post-assessment items, nearly all PSTs perceived area as the space inside or the amount of coverage for a two-dimensional region (K1a). When asked to define area (HW #1a), five PSTs described it as the space inside a two-dimensional region, two as the number of square units that cover a region, and two others as the face of a three-dimensional object. Only one confused area and volume. When provided with descriptions of different attributes of various objects (HW #2 and Test #4), all PSTs were able to identify when area was the associated attribute. Finally, on Follow-Up #1 and #2, four PSTs measured area using direct comparison (laying the shapes on top of each other), three counted the number of area units, and three compared the length and width dimensions. Thus, at least four PSTs were able to use direct comparison to measure area (S1) and at least five PSTs recognised area as the number of area units that cover the region (K1b).

When given a rectangle with dimensions labelled as five units and six units (HW #3), all PSTs used $length \times width$ to determine the area of the rectangle, while only one elaborated that she "would be able to fit 30 units in this [rectangle]." On HW #4, seven PSTs commented the area was 30 units because 30 squares fit in the rectangle (K1b). Of these, five commented that one might also use $length \times width$ to calculate the area. Three PSTs made no reference to the number of unit squares inside the rectangle and only used $length \times width$. Thus, approximately 70% of the PSTs understood that area may be measured by the number of square units that cover a region when the square units are shown (K1b), while at least half of the PSTs recognised multiplying $length \times width$ produces the same measurement as counting the unit squares (S1). However, none of the PSTs expanded on why $length \times width$ produces the number of unit squares. Furthermore, when asked to define area (HW #1a), five PSTs included $length \times width$ in their explanation along with a picture of a rectangle. It is not known whether the PSTs were trying to provide as much information as possible about area while knowing that the formula only applies to rectangles or whether the PSTs were equating area with $length \times width$. In conclusion, the PSTs held a rote understanding of this area formula.

Learning goal K1b includes an understanding of the mathematical advantages of using a *square* area unit. When asked to describe the pros and cons of various shapes for area units (Test #1), seven PSTs explained that a square tessellates, and one explained that it is easier to take fractions of a square. Five PSTs acknowledged that a circle unit leaves gaps between repetitions, while six explained how a trapezoid tessellates. Across all three shapes, seven PSTs strongly desired a resemblance between the shape being measured and the shape of the area unit as well as a need for boundedness (not wanting the area unit to cross the boundary of the region). The PSTs expressed concern that spaces would remain when using a unit that did not align with the edge of the shape measured (e.g., preferring circle area units to measure regions with curved edges and square area units to measure rectangular regions).

With regard to learning goal S2, the PSTs exhibited familiarity with some standard area units. When asked to provide units for measuring the area of various objects (HW #2), the PSTs provided US Customary units such as ft^2, in^2, and yd^2 and metric units such as m^2 and cm^2, possibly indicating some familiarity with the size of these units. However, we also suspect they relied upon their familiarity with linear units and then "squared" the units for area. In fact, at least $\frac{1}{3}$ of the PSTs appeared to interpret as a rule the need to square linear units rather than a result of having a square. For example, two PSTs stated that "area is given in units squared", while others at times overgeneralised the 'rule' (squaring units that were already area units) or failed to square linear units for area measures.

PSTs' Conceptions of Volume as Revealed on the Post-Assessments

Across the post-assessments, the PSTs interpreted volume as the space inside a three-dimensional object (K2a), providing similar descriptions when asked to define volume (HW #1b). However, four PSTs also included $length \times width \times height$ in their definitions, potentially equating volume with this formula. When provided with descriptions of different attributes of various objects (HW #2 and Test #4), all PSTs were able to identify when volume was the associated attribute. When asked to justify $length \times width \times height$ for a shoebox (Follow-Up #4), eight PSTs drew upon volume as the space inside a *three-dimensional* object (K2a): "If volume refers to 'how much of something fits into something else', it is necessary to look at all dimensions. A flat rectangle can't contain any liquid for example. In order for the box to have volume, it must be extended into a third dimension." However, none of the PSTs elaborated on how $length \times width \times height$ produces the number of cubic units. Thus, the PSTs revealed primarily a rote understanding of this volume formula.

Follow-Up #3 asked the PSTs to determine whether a cube, a cylinder, or a rectangular prism would serve as the best volume unit and why (K2b). Of the seven PSTs that correctly interpreted the item, three selected the cube without justification, three selected the cube describing how it tessellates space while the cylinder leaves gaps, and one explained that all three solids might work because they are all three-dimensional. Thus, between the volume station and the post-assessments, most of the PSTs recognised the need for volume units not to leave gaps, but only approximately half of the PSTs realised why cubes serve as an effective volume unit.

Across the post-assessments, the PSTs exhibited familiarity with some standard volume units (S2). When asked to provide appropriate units for measuring the volume of various objects (HW

#2), the PSTs provided US Customary units of tablespoons, fluid ounces, cups, pints, quarts, and gallons as well as metric units of mL, L, cm^3, and m^3. Finally, similar to area, $\frac{1}{3}$ of the PSTs demonstrated a rote understanding of why one cubes the units for volume measures, again stating as a rule that "volume is given in units cubed." On Test #5, three PSTs explained that the unit was cm^3 "because you have had 3 values for cm multiplied together." None of the PSTs alluded to 1 cm by 1 cm by 1 cm cubes nor appeared fully to understand how linear units constitute the cubic unit.

PSTs' Conceptions of Enduring Understandings on the Post-Assessments

The lesson experiment addressed EU1, EU2, EU3, and EU6 with respect to area and volume. On items requiring measurements to be matched with descriptions of physical objects (Test #6, HW #2), all PSTs matched area units with area objects (perhaps facilitated by the term "square" in the units) and seven PSTs matched volume units with volume objects (EU1a). However, we are unsure of the degree to which they explicitly understood that an area or volume unit has to have the same attribute as that being measured. For learning goal EU1b, the PSTs understood that directives to measure the area or volume were seeking a numerical response (HW #3 and #4, Test #7). The PSTs appeared to understand thoroughly learning goals EU1c, EU1d, EU2, and EU3 within the stations, so we did not directly assess these goals in the post-assessments, a decision we would adjust in hindsight as the stations primarily provided a group perspective on the PSTs' understandings.

With regard to learning goal EU6, all PSTs correctly completed Test #2, indicating at least a recall level of the PMI goals. Furthermore, on Test #3, all PSTs recognised that the activity involved working within a measurement system (Goal 3). In conclusion, the PSTs were able to order the various steps in the PMI, but questions remained. How well would the PSTs be able to identify activities representative of the other goals? Could the PSTs generate or find activities for the various goals? Did the PSTs understand the need for sequencing instruction in this fashion?

Steps Three and Four: Evaluating Hypotheses and Revising the Lesson

Hypothesis One

Having the PSTs complete either the area or volume station followed by a presentation on the other attribute appeared to enhance some of the PSTs' understandings of area and volume. The frequencies computed as part of Phase 1 allowed a pre- and post-lesson experiment comparison (Table 5). The PSTs appeared to enhance their understanding of area as the space within a two-dimensional region and of volume as the space inside a three-dimensional object. They also appeared to gain familiarity with using square units and cubic units respectively to measure area and volume as well as with various standard area and volume units. However, many of the PSTs still held rote interpretations of measurement formulas, were uncertain of the mathematical advantages of a *square* area unit and a *cubic* volume unit and failed to understand how area and volume units are constituted by length units. Thus, we found several sophisticated mathematical concepts need to be addressed with each attribute. We revised Hypothesis 1 as follows: *PSTs have varied background knowledge on area and volume, which should be evaluated through a pre-assessment and heeded when planning lesson activities. However, the learning goals with regard to area and volume are substantial enough to warrant all PSTs completing both the area and volume activities despite necessitating more class time*.

Table 5
PSTs' Understandings across the Lesson Experiment

Learning Goal	Specific Topic	Pre-*n*	Post-*n*
K1: Area	Area as the space covered by a two-dimensional object	4	All
	Confused area and volume	4	1
	Area is measured by counting square units	0	7
K2: Volume	Volume as how much a three-dimensional object can hold	5	All
	Associated volume with three-dimensions but used area language	3	None
	Volume is measured by counting cubic units	0	8
S1: Measure area and volume	Select area units for area and volume units for volume	---	All
S2: Units	Familiarity with many standard area and volume units	2	All

Hypothesis Two

Completing activities aligned with the PMI did enable the PSTs to meet some of the learning goals. To delineate further, we now reflect upon the affordances and shortcomings of each step in the stations. The claims that follow are based on evidence provided in the sections "PSTs' Work on the Area Station" and "PSTs' Work on the Volume Station".

Evaluation of the area station. An affordance of area Step 1 was highlighting the difference between length and area. While it may have been helpful to replace "area" with a synonym such as "coverage" to enrich the PSTs' perception of area, Step 1 did appear to support PSTs in understanding area as the space inside a two-dimensional region (K1a). Engaging PSTs in direct comparisons was an appropriate first step for broadening their formulaic views of area.

Area Step 2 enabled PSTs to determine area by counting square units (K1b and S1) as well as to experience the need for standard units (EU2) and the inverse relationship between a unit and a measurement (EU2ab). However, since this step provided *square* units, nowhere were PSTs required to consider the shape of the area unit. We did not expect the PSTs' preferences for resemblance and boundedness, which we have since learned that elementary students prefer as well (Lehrer, 2003). As recommended for elementary students, we therefore would insert an activity prior to Step 2. The PSTs would measure the area of irregular shapes using different shaped area units (e.g., triangles, trapezoids, circles, irregular units, etc.), and then consider the advantages and disadvantages of each for measuring area. The resulting discussion may address the need for area units to tessellate the plane and how to use fractional units as needed (especially convenient with square units). From this experience, we would add Learning Goal K1c (see Table 6) and adjust PMI Goal 2 to recommend investigations of the shapes of measurement units (see Figure 7).

Estimations such as those in the second half of Step 3 reverse the typical direction of measuring and thereby allow the unit to have meaning for the student (Bright, 1976), as we found with the PSTs. However, neither part of Step 3 addressed the PSTs' failure to view for example 1 m^2 as a 1 m by 1 m square or their lack of familiarity with the standard metric area units. Therefore, we recommend asking the PSTs to build standard area units. For example, PSTs may draw a square centimetre or use masking tape to mark off 1 m^2, attending to the linear dimensions of such squares. The instructor may then ask the PSTs to compare their constructions with the faces of the base-ten blocks or some overhead grids to realise how these tools may serve as "rulers" for area (Battista, 1982). We would then return to Step 3, still providing multiple copies of the standard units to emphasise direct measurement of area. For the first half of Step 3, the PSTs should be directed to predict the area measures before actually measuring, and for the second half of Step 3, the PSTs should measure to confirm their objects (Battista, 2003; Bright, 1976). In sum, we revised Learning Goal S2 to emphasise familiarity with the 'size and shape' of standard units (Table 6) and the third goal of the PMI to incorporate our recommended activities (Figure 7).

Finally, in completing area Step 4, the PSTs appeared to enhance their understanding of the order and comparative magnitude of the metric prefixes. However, the PSTs interpreted the area measurements as length measurements. We surmise that the revisions for Step 3 would assist with this issue. Thus, we would keep Step 4 as originally written, but we may also add some precursor questions such as "Use the base-ten blocks or a scale drawing to illustrate 6 square decimetres (6 dm^2)." This may assist PSTs with realising that 6 dm^2 consists of six squares and may be arranged in various ways as a rectangle, e.g. 2 dm by 3 dm.

Evaluation of the volume station. An affordance of Step 1 was its distinction between surface area and volume. Like the area station, it may be helpful to use a synonym for volume such as "capacity" or "space inside". Overall, however, Step 1 appeared to support the PSTs with viewing volume as the space inside a three-dimensional object (K2a).

Step 2 enabled the PSTs to determine volume in multiple ways (S1), to recognise the need for standard units with no gaps between them, and to experience the inverse relationship between the size of a unit and a measurement (EU1c, EU2ab). However, nowhere were the PSTs required to consider the implications of the solid selected for a volume unit. Thus, like the area station, we would insert an activity in which the PSTs measure the volume of irregular three-dimensional containers using different shaped volume units, addressing our new Learning Goal K2c (Table 6). Next, we would have PSTs complete Step 2 except provide two different sized cubes to use as the volume unit in measuring the cylinder.

Step 3 enabled the PSTs to directly measure volume (S1), to gain familiarity with a cubic centimetre (S2), and to appreciate the advantages of liquid units, which 'fill all the gaps' (K2b). However, this step was limited to cubic centimetres. Thus, we first recommend helping PSTs gain familiarity with more of the standard volume units by giving them unlabelled physical versions of such units (e.g., base-ten units for cm^3, 1-inch wooden cubes for in^3, etc.). Ask PSTs to measure the edges of such solids and determine the standard volume units represented. For standard volume units difficult to locate in physical form, PSTs may draw them to scale on isometric grid paper or construct them out of cardboard. PSTs may then be asked to directly measure various solids using multiple copies of the standard units. Finally, similar to the area station, we would conclude this step by asking PSTs to find objects with given volume measurements.

On Step 4, most PSTs realised that 1 L is equivalent to 1 dm^3 and that 1 cm^3 is equivalent to 1 mL using the $\frac{1}{1000}$ scale (S2). However, we are uncertain of how thoroughly the PSTs perceived 1000 cm^3 in 1 dm^3 and 1 cm^3 as holding 1 mL. Thus, we recommend this step begin by asking PSTs to place base-ten units (cm^3) inside a hollow base-ten cube (dm^3). Then, after PSTs pour the 1 L into the base-ten cube and surmise that 1 mL is equivalent to 1 cm^3, we would have PSTs pour 1 mL of water into a hollow base-ten unit (cm^3). The original step also did not require the PSTs to develop familiarity with other liquid volume units, so an instructor may want to follow Step 4 with activities using such units. Finally, due to the affordances of Guessing the Unit in area, we recommend a parallel activity in volume.

Measurement formulas. Our intent for the lesson experiment was to enhance the PSTs' conceptual understandings of area and volume (K1a, K2a) and then to address area and volume formulas in future lessons. However, we inadvertently included some measurements that were more easily computed with a formula than by covering or filling (e.g., surface area of the front of a door). The PSTs determined such measurements using their prior knowledge of measurement formulas, yet they exhibited multiple challenges while doing so. Having witnessed these challenges, we returned to our lesson design as well as further literature about PSTs' and grade K-12 students' challenges with such formulas. As a result, we enhanced our understandings of the difficulties PSTs' face with regard to the formulas $length \times width$ and $length \times width \times height$. While we acknowledge there are several measurement formulas PSTs should investigate, these two formulas emerged as most relevant for the stations, so we recommend addressing at least these two within the associated lesson.

Prior to the lesson experiment, we were familiar with the understandings necessary for making sense of the area formula for rectangles and intended to address such understandings in later lessons. Specifically, to think about area as a product of lengths, one first has to structure the rectangle as an array of rows and columns (Lehrer, Jaslow, & Curtis, 2003; Battista, 2003). Next, one joins the individual units within a row into a composite unit and then repeats down the composite unit a number of times corresponding with the width (Battista, 2003; Stephan & Clements, 2003). Many researchers have found that K-12 students as well as PSTs may not understand how $length \times width$ correlates with this iterative process (e.g., Battista, Clements, Arnoff, Battista, & Van Auken Borrow, 1998; Lehrer, Jaslow, & Curtis, 2003; Simon & Blume, 1994). New to us however was that even when PSTs do internalise this process, they still often fail to see how the linear units constitute the individual area units (Simon & Blume), necessary for the mathematical convention of defining area units by length units. Thus, a full understanding of area extends beyond just counting individual area units and requires coordinating two dimensions (Lehrer, Jaslow, & Curtis).

Our further reading of the literature then also revealed information about teachers' perceptions of area and downfalls of using manipulatives for covering areas. First, Outhred and McPhail (2000) found that teachers tend to conceive area measurement as only counting the area units that cover a region. This counting is detrimental when students have to determine the area of irregular shapes that require subdividing a region. Second, manipulatives such as physical tiles mask the unit structure because they do not encourage students to iterate a composite unit as they may be counted one by one when they are laid down (Outhred, Mitchelmore, McPhail, & Gould, 2003; Stephan & Clements, 2003). Thus, while one of the strengths of area Step 3 is directly measuring area with multiple standard area units, it allows the PSTs to remain at a counting perspective of area. After Step 3, we would direct PSTs to measure the surface area of a large object, such as a whiteboard, while only providing them with *one* copy of the area unit, perhaps one ft^2. PSTs likely will use the ft^2 to measure the length and height dimensions of the whiteboard and then multiply these values. The instructor may then pose the following questions to encourage PSTs to recognise the process of iterating a composite unit: "With this process, the corner square is counted twice (once when measuring the length and once when measuring the height). Is that a problem? Why or why not (Battista, 2003; Simon & Blume, 1994)? Why do we multiply our length and height measurements rather than for example add them?"

Next, to address what we have since learned about PSTs' understandings of how the linear units constitute the area unit, we would pose the Stick Problem: *Imagine Shannon and Olivia work together to measure the area of a rectangle. Shannon measures the length while Olivia measures the width. Each measures their dimension with a stick; however, the sticks are of different lengths. Shannon says, "The length is four of my sticks." Olivia says, "The width is five of my sticks." What have they learned about the area of the rectangle?* (Adapted from Simon & Blume, 1994, p. 487). Note the problem only provides linear units and results in a non-square area unit, drawing attention to how each linear measure contributes to the area unit (the area is 20 rectangular units, each with the length of Shannon's stick and the width of Olivia's stick). Finally, we would provide PSTs with a 10 x 20 cm rectangle and a single 2 x 5 cm rectangular unit (both without dimensions labelled) and direct them to measure the rectangle with the provided area unit. Some PSTs will likely maintain the orientation of the rectangle when measuring length and width (resulting in a measurement of 20 2 x 5 rectangular units), while some may not (resulting in either 8 5 x 5 square units or 50 2 x 2 square units). The instructor may then ask the PSTs to compare the numerical quantities of the resulting measurements and to consider the area unit for each. Eventually, the instructor may conclude with a discussion of how linear units constitute the area unit.

Misconceptions also arose with regard to the volume formula and how linear units constitute a volume unit. To assist PSTs with developing such understandings, we would mirror the activities recommended for area but adjusted for the third dimension of volume, e.g., discuss whether the corner cube is counted three times, add a third stick of different length, and discuss measurements and volume units when the unit is oriented in different ways. If the stations are supplemented with the aforementioned activities, we recommend adding Learning Goals K1d, K2d, S1b, S1c, and S1d (Table 6). In addition, because of our need to differentiate more clearly between direct measurement and indirect measurement, we propose revising PMI Goals 3 and 4 (Figure 7).

Table 6
Revised Knowledge and Skills for the Area and Volume Lesson Experiment (additions in italics)

Knowledge (K)	Skills (S)
K1a: Area is the space inside a two-dimensional region. K1b: Area is measured by the number of square units that cover the region. *K1c: An area unit has to tessellate the plane. As partial units are used when area units do not align with the edge of the measured shape, squares serve as the normative mathematical unit for area.* *K1d: Understand how the measurement formula $length \times width$ structures a rectangular region as an array as well as how the linear dimensions determine the number of iterations of the composite unit.* K2a: Volume is the space inside a three-dimensional solid. K2b: Volume is measured by the number of cubic units (solid or liquid) that fill the space. *K2c: A volume unit has to tessellate space. As partial units are used when volume units do not align with the outside edge of the measured solid, cubes serve as the normative mathematical unit for volume.* *K2d: Understand how the measurement formula $length \times width \times height$ structures a rectangular prism as an array of cubes as well as how the linear dimensions determine the number of iterations of the composite unit.*	S1a: Compare, order, and directly measure area and volume using nonstandard and standard units. *S1b: Differentiate between length, area, and volume units.* *S1c: Constitute area and volume units from linear units.* *S1d: Indirectly measure the area of rectangles and the volume of rectangular prims using the measurement formulas $l \times w$ and $l \times w \times h$, respectively.* S2: Develop familiarity *with the size and shape of* standard units for area and volume (e.g., 1 m^2 as a 1 m by 1 m square and 1 m^3 as a 1 m by 1 m by 1 m cube)

In conclusion, for Hypothesis 2, completing stations in sequences aligned with the PMI appeared to enhance some of the PSTs' understandings of area and volume. Thus, we would maintain Hypothesis 2 as stated. However, we also found that the PMI lacked specific guidance about some of the intricacies involved and recommend that Hypothesis 2 be undertaken with our Revised PMI and Learning Goals.

Revised Plan for Measurement Instruction (revisions in italics)

Goal 1: Students will understand the attribute to be measured.

- Have students make comparisons of the attribute with different objects. For example: Which is longer/shorter? heavier/lighter? holds more/less? Use direct comparisons whenever possible.

Goal 2: Students recognise the need for and use units to produce a measure.

- Present students with objects to compare and measure for which direct comparison is no longer feasible to generate the need for and understanding of a unit. Have students measure physical models of the attribute using first nonstandard and then standard units. Include activities where students have multiple copies of the unit available and where students have only one copy of the unit available.
- **Comment:** The teacher can help students understand the need for a common unit by asking them to measure a single object with different sized units.
- **Comment:** *The teacher may help students appreciate the shape of a standard unit by allowing students to experience the advantages and disadvantages of other shaped units.*

Goal 3: **Students will work within a measurement system, including its standard units and common measuring tools, to directly measure attributes.**

- Have students make their own measuring instruments with informal units and then compare how those are measuring in the same way that standard instruments do;
- *Engage students in predicting and directly* measuring objects with standard units;
- Plan activities for students to develop familiarity with *the size and shape* of standard units, *such as constructing standard units, finding objects with a given measure, and guessing the unit for given objects and measurements.*

Goal 4: **Students will use measurement formulas with understanding and flexibility to indirectly measure attributes.**

Figure 7. Revised Plan for Measurement Instruction.

Hypothesis Three

Hypothesis 3 outlined ways to support PSTs in understanding pedagogical implications of the PMI. The PSTs did experience the PMI with angle and length, and the instructor referred back to these activities when introducing the PMI. However, in the group presentations, neither group related the station steps to the PMI goals. Thus, we would revise the presentation directions to explicitly direct PSTs to align the steps in the stations with the PMI goals. We also would implement a follow-up activity in which PSTs align various measurement activities with the PMI. Indeed, the instructor facilitated such a lesson in her class a few days after the lesson experiment. Finally, the PSTs were able to order the various steps in the PMI, but questions remained. Therefore, we recommend incorporating more assessment items in this regard as well as enhancing Hypothesis 3 as follows: *The PSTs will better understand the pedagogical implications of the PMI if they experience the progression as learners themselves, have illustrative activities to refer to when introduced to the plan, are asked to interpret and eventually identify/create measuring activities with respect to the plan, and are required to consider the rationales for proceeding measurement instruction in this fashion through readings, discussions, and activities.*

Conclusion

The purpose of this project was to further our knowledge and abilities as mathematics teacher educators by critically reflecting on our practice of teaching prospective teachers. Specifically, we used a lesson experiment to examine how our area and volume lesson affected the PSTs' understandings. Our intent with this paper is to share what we learned about how instruction may support PSTs' understandings of area and volume as well as our experience with the lesson experiment process. The sections above articulate our findings with regard to how instruction affected the PSTs' understandings. Without the lesson experiment approach, we doubt that we would have recognised these ideas about instruction and understanding. Now, we turn our attention to our experience with the lesson experiment.

We learned three things about conducting a lesson experiment. First, while we were used to stating learning goals and planning instruction accordingly, we were not used to stating our hypotheses for instruction. We found spelling out such hypotheses a crucial part of the process. Without doing so, the lesson experiment may have easily deteriorated into a simple description of how to tweak the lesson without specific rationales for teaching decisions, lacking transparency for us or other teacher educators considering how the lesson may translate to other contexts. Second, we learned to be forgiving towards ourselves. While we thought we had sufficiently determined our learning goals, planned the instruction, and designed the assessment items, the lesson experiment revealed improvements to all of the above. At first, we felt that we had failed in our preparation. In hindsight however, we realise this is an intentional part of the process. In practice, most mathematics teacher educators are not able to be familiar with all literature or knowledgeable about all aspects of a topic before they have to teach it. Furthermore, personal experience may be necessary before one is prepared to internalise or adapt information provided in the literature for a particular context with PSTs. Fortunately, one of the main purposes of a lesson experiment is to help mathematics teacher educators learn from their teaching practice and thereby improve their instruction over time.

Third, while we feel that we made some real-time improvements to our area and volume instruction, our lesson experiment adjustments appear to be more thorough, accurate, and empowering of PSTs' understandings. In real-time, prior to our formal analysis, the instructor did make some adjustments after the area and volume lesson. She incorporated a follow-up activity in which the PSTs sorted length activities by the PMI goals, and she had the PSTs use cylinders, rectangular prisms, and cubes to fill a shoebox and compare which was the better unit for measuring volume. However, before our lesson experiment analysis, our conclusions only consisted of paraphrases of the PSTs' ideas, failed to quantify frequencies across the PSTs' understandings, and missed many aspects of the PSTs' thinking. For example, we failed to see the PSTs' desire for resemblance and boundedness with area units, their lack of visualisation of the size of area and volume units, and their over-reliance on symbolic methods. Furthermore, the lesson experiment analysis led us to seek further literature on children's and PSTs' thinking about area and volume and better prepared us to interpret and adapt to our context the additional literature we found. For example, we identified instructional recommendations for challenges that we did not recognise in real-time (e.g., the Stick Problem). While the lesson experiment process takes more time and may not affect current instruction, the overall gains in how instruction impacts PSTs' understandings may be incorporated in future courses for many years to come, helping mathematics teacher educators build their knowledge and abilities over their professional lifespans. The lesson experiment process thus affords more than just a casual reflection on teaching a lesson.

In sharing our lesson experiment with other teacher educators, we offer a few recommendations. First, we feel the best way to learn about the lesson experiment process is to try it. We recommend teacher educators do what they can in preparing for the lesson and then undertake the process. We feel doing so greatly enhances one's understanding of the lesson experiment process and will undoubtedly increase one's knowledge of how instruction is affecting the understandings of PSTs. In addition, we want to encourage teacher educators to disseminate their results, reporting not only their findings, but 'how they learned'. "This will allow such research to contribute to greater theoretical understanding about mathematics teacher education and mathematics teacher educator learning and ultimately to the improvement of practice" (Chapman, 2008, p. 132).

In closing, lesson experiments need to undergo iterative processes of implementation and revision, including our lesson experiment here. Successive implementations will verify whether our recommendations are truly improvements or just changes. We hope however that our first iteration provides a model for mathematics teacher educators striving to learn from their practice of teaching teachers.

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