Pre-service Teachers' Understanding of Fraction Multiplication, Representational Knowledge, and Computational Skills

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Despite the importance of teacher fractional knowledge, there are several areas of teacher understanding that are not well understood. The purpose of this study was to characterise profiles of pre-service teachers' (PSTs) mathematical competence on the topic of fraction multiplication by examining PSTs' understanding of multiplication of fractions in three different contexts. Analyses of 60 PSTs' written responses revealed that there are distinct gradations of competency, ranging from the PSTs who were unable to solve a given problem in any context to those capable of flexibly portraying understanding of fraction multiplication in three contexts. Most PSTs who recognised the word problem as a multiplication of fractions were able to explain their thinking using graphical representations. However, we also observed various types of errors PSTs made in representing the word problem in graphical representations and translating it to a correct multiplication expression. These findings offer descriptors of how the PSTs understand fraction multiplication in different contexts and provide information for the design of interventions in teacher education. One objective would be to support recognition of the connectivity of fraction multiplication in different contexts.

Keywords · Teacher knowledge · Fraction multiplication · Pre-service teachers · Hypothetical learning trajectory · Content knowledge

Introduction

Since Shulman (1987) coined the notion of content knowledge and pedagogical content knowledge, many researchers have investigated what teachers know about mathematics and teaching mathematics and how they know it. They have reported teachers' insufficient knowledge about teaching mathematics, in particular, elementary in-service and pre-service teachers' lack of understanding about whole numbers, fractions, and fraction operations (e.g., Depaepe et al., 2015; Hill, Rowan, & Ball, 2005; Krauss, Baumert, & Blum, 2008; Mack, 2001; Newton, 2008; Senk et al., 2012; Son, 2013; Son, 2016a, 2016b; Son & Crespo, 2009; Son & Sinclaire, 2010; Steffe, 2003; Tirosh, 2000). This line of research studies suggests that teachers should develop a profound understanding of fundamental mathematics (Ma, 1999) and mathematical knowledge for teaching (e.g., Ball, Thames, & Phelps, 2008), which encompasses knowledge that teachers use in teaching practice, such as using curricular materials appropriately, selecting and using effective representations, deftly assessing students' work, and providing appropriate remediation. Teachers' capability to use different representations of mathematical ideas is considered to be an important area of mathematical knowledge to develop in order to provide meaningful learning opportunities for students (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010; National Research Council [NRC], 2003). In particular, transitions between and within each representation can help students develop new concepts flexibly and efficiently (e.g., contextual, visual, verbal, physical, and symbolic representational forms) (Izsák, 2003; Lamon, 2001; National Council of Teachers of Mathematics [NCTM], 2000; NRC, 2003; Whitin & Whitin, 2012; Zazkis & Liljedahl, 2004). Empson (2002) highlighted that "the key in fraction instruction is to pose tasks that elicit a variety of strategies and representations" (p. 39) and that representational models used by teachers (e.g., pizzas, fraction tiles, number lines, and fraction bars) engaged and facilitated students' learning of initial fraction knowledge (Cramer, Post, & delMas, 2002; Cramer & Wyberg, 2009).

However, despite a great deal of research on teacher knowledge, several questions still remain unanswered in the current literature. For example, how is PSTs' ability to solve word problems related to their ability to use representations and complete computations correctly? There is a relative lack of attention in current literature to PSTs' ability to translate from one mode of representation to another and the possible difficulties PSTs have in connecting different modes of representation (e.g., connecting pictorial representation to computational procedures that involve fraction multiplication) (Hackenberg & Tillema, 2009).

The purpose of this study was to examine what pattern, if any, exists in the pre-service primary and middle-grade teachers' (PSTs) abilities to solve problems involving multiplication of fractions when such multiplication is presented in three different forms of representation: (1) a word problem format, (2) a purely symbolic notation format, and (3) a format requiring the use of visual representation. By investigating the relationship between PSTs' ability to solve word problems and their ability to use representations and complete computations correctly, we also intended to characterise profiles of the PSTs' mathematical competence on the topic of fraction multiplication. To do so, we explored how PSTs perform differently in three different modes and investigated which form is more difficult for them to use when solving fraction multiplication problems. By offering descriptors of how the PSTs understand fraction multiplication in different contexts, we intended to provide information for the design of interventions in teacher education. Research questions that guided this study are as follows:

1. How do PSTs perform differently when multiplication of fractions is presented in three different contexts: in a word problem format, in a purely symbolic notation format, and in a format requiring the use of visual representation?
    (a) What kinds of graphical representations do they use to justify their answers?
    (b) What concepts do PSTs use in explaining and justifying the meaning behind their computation steps?
    (c) What method(s) do they use to solve a multiplication of fractions problem in a purely symbolic notation format?
2. What pattern, if any, exists in the PSTs' abilities to solve multiplication of fractions problems in three different modes?

Kilpatrick, Swafford, and Findell (2001) define mathematical proficiency as successful mathematics learning. They construct five components of mathematical proficiency: (1) conceptual understanding (an integrated and functional grasp of mathematical ideas); (2) procedural fluency (knowledge of procedures, knowledge of when and how to use them appropriately, flexibly, accurately, and efficiently); (3) strategies competence (the ability to formulate mathematical problems, represent them, and solve them); (4) adaptive reasoning (the capacity to think logically about the relationships among concepts and situations); and (5) productive disposition (the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, and to see oneself as an effective learner and doer of mathematics) (p. 116). The research questions asked in this study can assess the first four components of mathematical proficiency. For example, conceptual understanding can be assessed in all three sub-questions in the first research question. Procedural fluency can be assessed in the third sub-question of the first research question. Adaptive reasoning can be assessed in the second sub-question. Strategic competence can be assessed in the first sub-question.

Theoretical Background

Unit Structures Entailed in Understanding Fraction Multiplication

Many multiplicative situations require an appropriate conceptualisation of the whole (unit) before the situation can be understood and a solution procedure can be implemented (Hiebert & Behr, 1988). In the middle years of school mathematics, a fraction is typically interpreted as a part-whole relationship, that is, partitioning an object or a set of objects into equal parts. A whole is cut into n slices; each slice is encoded as 1/n; and if we refer to several slices (k), that idea is encoded as k/n. The idea of one as a whole is a basic feature. Although the repeated-addition interpretation of multiplication is ordinarily the first one that students encounter, when operations on whole numbers are expanded to include rational numbers, the meaning of multiplication must also be expanded (Son, 2012). The models representing multiplication can be expanded from repeated addition to include a fractional part of a whole. For example, the following problems can be solved by repeated addition: At the supermarket, potatoes were bagged in 3/4-pound bags. Mum bought 3 bags of potatoes. How many pounds of potatoes did Mum buy? To find the total amount of potatoes, we need to add 3/4 three times (3/4 + 3/4 + 3/4). Situations such as this can be modelled by the repeated-addition interpretation since the link between multiplication and addition is clear; that is, the problem deals with three equal-sized groups wherein each group consists of one 3/4-pound bag of potatoes. However, there are other situations that can be represented by multiplication but that do not involve direct repeated-addition situations, as shown in the following example:

> Julie bought 4/5 of a yard of material for her class project. Later, she found that she needed only 3/4 of the material she bought. How much material did Julie use for her project?

Students should know that 4/5 of a yard becomes the whole in this problem. Students need to find 3/4 of this whole. In this problem, students must realise that the whole is itself part of a unit. The fractions 4/5 and 3/4 refer to two different wholes (referent units, that are the whole used in each operation). When the correct referent units are identified, determining what fractional part of which referent unit to consider becomes clearer. Thus, it is necessary to flexibly coordinate not just two (parts of a whole) but three levels of units (parts of parts of a whole) in order to fully understand the meaning and procedures involved in the multiplication of fractions.

Several researchers have examined students' understanding of the levels of nested units. A series of studies conducted by Steffe and Olive (Olive, 1999; Olive & Steffe, 2002; Steffe, 2002, 2003, 2004) claim that the differences in students' ability to coordinate levels of units led to significant differences in the fraction schemes and operations (Empson, 2002; Empson, Junk, Dominguez, & Turner, 2006). For example, learning to partition another partition (i.e., recursive partitioning) is critical to reasoning about the process involved in fraction multiplication.

Understanding fractions of fractions as parts of parts establishes three-level unit structures when interpreting parts of parts against the original whole. Steffe (2003) presents three states (or levels) of mental operation that require students to conceptualise and reconceptualise unit structures for fraction multiplication.

In a similar vein, Mack (1998, 2000) points out that although multiplication can be viewed in several ways, such as repeated addition or arrays, many situations with fractions involve taking a part of a part of a whole. She describes how teachers might build a foundation for understanding the multiplication conception of fractions to fifth-grade students in a meaningful way. To develop students' understanding of multiplication of fractions, she posed the following problems in order:

- Equal-sharing situations (e.g., sharing 10 cookies among 4 people)
- Finding a fraction of a whole-number amount (e.g., 2/3 of 12)
- Taking a part of a part of a whole (e.g., 1/4 of 4/5)
- Taking a part of a part of a whole; the denominator of the first fraction is a multiple of the numerator of the second fraction (e.g., 3/4 of 2/3)
- Taking a part of a part of a whole; the denominator of the first fraction is a factor of the numerator of the second fraction (e.g., 2/3 of 9/10)
- Taking a part of a part of a whole; the greatest common factor of the denominator of the first fraction and the numerator of the second fraction is 1 (e.g., 3/4 of 7/8).

Mack (2001) examined how students could build on their limited conception of multiplication with whole number to develop an understanding of the multiplication of fractions. The results show that when students thought of fractions only in terms of the number of parts where each part represented an independent whole-number quantity (e.g., 3/4 means "three of four parts"), they were unable to reconceptualise and partition units in circumstances when each part consisted of more than one element (e.g., finding 2/3 of 9/10 or 3/4 of 7/8). However, when students focused on the number of parts and considered the fractional amount that each part represented of a unit (e.g., 3/4 means three fourths of one whole or three units of one fourth), they were able to reconceptualise and partition units in a variety of ways. Based on prior research described above, we have conceptualised the following possible trajectory for the development of representational ability involving fraction multiplication. This trajectory is used to characterise profiles of the PSTs' representational competence involving fraction multiplication.

- Stage 1: Partitioning. The student conceives of partitioning only in relation to a unitary whole; he or she is not able to apply partitioning to parts of wholes.
- Stage 2: Coordination between Parts and Parts. The student is capable of applying partitioning to parts of wholes and decompose an initial unit into a unit of units (i.e., recursive partitioning) but is not able to identify a referent unit.
- Stage 3: Identification of referent units. The student is able to partition a partition in service of a non-partitioning goal and decompose an initial unit into a unit of unit structure. Students in this stage are able to relate parts of parts back to the original whole.

The Type of Models for Fractions

There is substantial evidence to suggest that the use of models/representations in tasks is important, and that models are useful when meaningful connections are made between the ideas being represented and the ideas that were intended to be represented (Cramer & Henry, 2002; Zazkis & Liljedahl, 2004). In this study, we use the two terms models and representations interchangeably. The concept of fractions and operations involving fractions can be represented by various physical or pictorial representations. For example, Van de Walle, Karp, and Bay-

Williams (2010) provide three types of models for fractions: area or region models, length or linear measurement models, and set models. Popular area or region models include circular "pie" pieces, rectangular regions, pattern blocks, and paper folding. Fraction strips, number lines, and line segment drawings can be used as length or measurement models. The common set model uses counters.

Wu (2001) emphasises the use of pictorial representations as a medium to guide teachers and students to reason about situations, discover patterns, seek ways to validate their own thinking, and convince others that their thinking is correct. Various representations/models can be used to help clarify ideas that may be not evident when presented in symbolic-only format. The area model, which represents fractions as parts of an area or region, has some advantages in that it is a familiar way for students to interpret the multiplication of whole numbers and the relative size of a part to a whole is emphasised (Graeber & Tanenhaus, 1993; Thompson & Saldanha, 2003; Van de Walle et al., 2010). In length models, measurements or lengths are compared when a certain length can represent the whole. The length model helps students connect fractions to measures on a ruler, thus providing a context for students to see fractions as numbers. In particular, the number line model can illustrate a fraction's relative size compared to other numbers and show that there is always another fraction between any two given fractions (Lannin, Chval, & Jones, 2013). Set models use a set of objects, having subsets of the whole set represent a fractional part of the whole. Set models have advantages when making connections with ratio concepts and real-life applications. However, using the entire set of objects to represent the whole is often confusing for students to grasp (Van de Walle et al., 2010).

Aksu (1997), who investigated student performance in three different settings, revealed that when fractions were given in the form of word problems, multiplication of fractions was the most difficult operation. He investigated how student performance differed when fractions were presented in different contexts. A test on fractions involving three parts—a concept test (understanding the meaning of fractions), an operations test (computations with fractions), and a problem-solving test (solving word problems involving fractions)—was administered to 155 sixth-grade students. The results revealed that student performance was highest on the operations test and lowest on the problem-solving test. No differences in performance of the four operations were observed when fractions were presented as computations. When fractions were given in the form of word problems, however, addition was the easiest operation to perform and multiplication was the most difficult.

In a study of multiplication of fractions, Izsák (2008) claims that the unit structures teachers employed shaped the purposes for which they used drawings when teaching fraction multiplication, as shown in the following cases:

- A teacher who reasoned primarily with just two levels of units and used drawings to illustrate solutions;
- A teacher who evidenced more consistent attention to three-level structures and used drawings to infer a computation method;
- A teacher who possesses the ability to explicitly produce three levels of units;
- A teacher who possesses the ability to explicitly produce three levels of unit structures flexibly so as to compare alternative approaches for producing three-level structures.

This work implies that teachers' or students' ability to use representation is closely related to their conceptual understanding of the unit structures involved in fraction multiplications, which can be a reasonable indicator for conceptual understanding of fraction multiplication.

Teachers' Understanding of Fraction Multiplication

There is no doubt that teacher knowledge is an important element in student learning. Shulman (1986) claims that a personal understanding of the subject matter may be necessary, but it is not a sufficient condition for being able to teach; teachers must find ways to communicate knowledge to others. Teachers must have two types of subject matter knowledge: knowledge of the subject field and knowledge of how to help their students come to understand the field. However, many researchers have shown that both in-service and pre-service teachers have a limited understanding of multiplication of fractions and have procedure-oriented knowledge.

Bezuk and Armstrong (1995) reported that teachers have a limited understanding of the meaning of multiplication of fractions. They designed a test to gather information about the computational and conceptual knowledge of the teachers, which included the following word problem:

> Three-fourths of the seventh grade class went to the football game. Of the ones who went to the game, one-third went by car. What part of all of the seventh graders went to the game by car? Explain your answer.

They reported that most of the teachers recognised the word problem as one where multiplication of fractions could be applied but had difficulty explaining their thinking. In particular, these teachers lacked the language with which to explain their thinking, while some revealed confusion in their attempts to draw diagrams to match the algorithmic solutions.

Teachers' as well as PSTs' limited content knowledge and pedagogical content knowledge on rational numbers has continuously been reported. For example, a tendency of generalising rules regarding discreteness of natural numbers to rational numbers (e.g., stating the next number after 3/5 is 4/5) is found in the majority of Finnish PSTs (Merenluoto & Lehtinen, 2002). A study of U.S. PSTs' content knowledge of fractions also reports their limited and fragmented knowledge, as shown in their misapplication of algorithms and modest flexibility when operating on fractions (Newton, 2008). Depaepe et al. (2015) examined and compared content knowledge and pedagogical content knowledge of elementary PSTs (future general classroom teachers) and secondary PSTs (future subject-specific classroom teachers) focusing on rational numbers. Their results showed that secondary PSTs outperformed elementary PSTs on content knowledge, but there was no significant difference in pedagogical content knowledge between the two groups of PSTs, confirming the claim that teachers' content knowledge is a necessary but not sufficient condition for pedagogical content knowledge.

By adapting items and formats used with students and in-service teachers in the previous studies (Aksu, 1997; Bezuk & Armstrong, 1995), this study examined PSTs' performance on multiplication of fractions. It intends to compare findings from previous studies with teachers and students by focusing on PSTs' work in the three different settings (i.e., word problem, purely numerical problem, graphic representation). By identifying the difficulties PSTs have in different contexts, this study aimed at extending the current literature on teacher knowledge and at providing practical implications for teacher educators to consider regarding their teacher preparation programs.

Methodology

Participants

The participants of this study included 60 PSTs who were in their second year of a teacher preparation program at a large South-eastern university in the United States. They ranged in age from 22 to 27 years, with 54 females and 6 males. They spent one semester taking a mathematics content course that was designed to help PSTs deepen their understanding of numbers and operations used in primary and middle school mathematics with respect to the meaning and use of representations. All three classes of the mathematics content course were taught by the first author during the 2012-2013 academic year for approximately 14 weeks.

During each lesson, PSTs were involved in learning the fundamental ideas for school mathematics. The main focus of instruction was on developing a deeper understanding of primary and middle years of school mathematics. It was important to develop the PSTs' understanding in terms of "why" certain mathematical procedures work and not just in terms of knowing how to calculate with whole numbers and fractions. Portions of three 3-hour sessions were devoted to fraction multiplication. In the first session, participants were invited to discuss why and when fraction multiplication was used. They were also asked to solve several different multiplication problems and discuss their strategies, which took about one hour. Three full hours in the second session were devoted to learning each type of fraction multiplication with different presentations and practicing those strategies. Students practiced these strategies individually at first, and then with their group members to offer different ways of presenting problems. Practice problems were also given to the participants as homework. In the third session, participants discussed the homework problems and discussed the relationship between fraction multiplication and fraction division. Approximately one hour was devoted to fraction multiplication in the third session. The main instrument for this study, which will be discussed next, was administrated to all the PSTs in the entire session in three mathematics methods classes towards the end of the semester. This report includes only the data of the 60 PSTs who signed the consent form and completed the study.

Instrument

A written task based on previous studies (Bezuk & Armstrong, 1995; Ma, 1999) was designed and used for this study. The task for this study was comprised of three questions that took about 15-20 minutes for the participants to complete. No time limit was imposed. Figure 1 shows the instrument used for the study. Questions 1 and 3 were devised to assess participants' content knowledge (i.e., specialised content knowledge). Question 2 was developed to assess their pedagogical strategies (i.e., knowledge of content and teaching) by asking them to provide possible teaching approaches to use to teach a word problem involving fraction multiplication. Each individual participant provided written responses.

Question 1: How would you solve the word problem below?
Three-fourths of the seventh grade class went to the football game. Of the ones who went to the game, one-third went by car. What part of all of the seventh graders went to the game by car?
Question 2: Imagine that you are teaching the multiplication of fractions to fifth-grade students. How would you use the word problem above? To make this meaningful for students, you can use graphical representations such as pictures, drawings, etc.

Explain this problem to fifth-grade students using some representations.
What concepts do you think students should know to solve this problem? Students already know that $\frac{3}{4}$ means "three of four parts." Describe and specify as much as you can.

Question 3: People seem to have different approaches to solving problems involving multiplication of fractions. How do you solve the problem below? Solve the problem. And justify your answer with multiple representations such as verbal representations, drawing, fraction bar, etc.

$\frac{3}{4} \times \frac{2}{3} = ?$

Figure 1. Instrument for the study

Data Collection and Analysis

The task described above was administered to three content classes at the end of the semester as a final exam in the 2012–2013 academic year. Data analysis involved four processes: (1) an initial reading of each PST's response; (2) identifying correctness of the responses; (3) exploring the type of models, explanation, and strategies used; and (4) interpreting the data quantitatively and qualitatively (Creswell, 1994). For example, once the responses for the first question were identified as correct, the responses were analysed by looking at the types of representations the PSTs used to solve the problem. More specifically, the PSTs used different types of representations such as graphic, symbolic, or verbal forms using written descriptions. Regardless of type of model/representation, their responses were counted as correct if they recognised the word problem as involving fraction multiplication and produced a correct answer. Once frequencies and percentages were obtained, patterns were explored with respect to teacher knowledge (i.e., competence level).

Results

Analyses of data showed that only 40% of the participants correctly recognised a word problem as a context that involved the multiplication of fractions, used representations correctly, and translated it to a correct multiplication expression. These PSTs used various graphical representations including linear length models (e.g., fraction bars, number lines), area models (e.g., rectangles), and set models (e.g., counters). However, only a few PSTs recognised the underlying concept of fraction multiplication as finding portions of portions, while most of them relied on a standard traditional algorithm. In the following section, categories representing PSTs' ability to solve problems for the multiplication of fractions in the three contexts were presented, followed by detailed descriptions of the difficulties PSTs experienced within each context.

PSTs' Performance and Difficulty with Fraction Multiplication in Three Contexts

Eight types of performance were identified based on the correctness of PSTs' responses in the three problem contexts (see Table 1). Type 1 represents a case in which PSTs solved the given multiplication of fraction problems correctly in all three formats–in a word problem format, in a purely symbolic notation format, and in a format requiring the use of visual representations. Types 2, 3, and 5 represent a case wherein PSTs make an error(s) in one of the three problem contexts. In contrast, types 4, 6, and 7 describe a case wherein PSTs solved the multiplication of fraction problems correctly in only one context. For example, PSTs in type 4 solved the word problem (Question 1 shown in Table 1) correctly, but did not use a correct representation in Question 2 with an incorrect method in Question 3. Type 8 represents a case wherein a PST failed to solve the given multiplication of fraction problems in all three formats, which is the lowest level of mathematical competency of PSTs. Table 1 shows the frequency of PSTs in each type and the corresponding percentages, which were rounded to the nearest percent to add up to 100%.

Table 1
The frequencies and percentages of PSTs in eight types

	Word problem (Question 1)	Representation (Question 2)	Computation (Question 3)	Frequency of PSTs (Percentage)
Type 1	correct	correct	correct	24 (40%)
Type 2	correct	correct	incorrect	9 (15%)
Type 3	correct	*incorrect*	correct	3 (5%)
Type 4	correct	*incorrect*	incorrect	1 (2%)
Type 5	*incorrect*	correct	correct	10 (16%)
Type 6	*incorrect*	correct	incorrect	1 (2%)
Type 7	*incorrect*	incorrect	correct	3 (5%)
Type 8	*incorrect*	incorrect	incorrect	9 (15%)
Total	37 (62%)	44 (73%)	40 (66%)	60 (100 %)

Only 40% (24 out of 60 PSTs) successfully solved the fraction multiplication problem in three contexts (Type 1); 15% of PSTs did not solve the problem correctly in any context (Type 8). Consistent with previous studies (Hill, Rowan, & Ball, 2005; Krauss et al., 2008; Mack, 2001; Newton, 2008; Son, 2013; Son, 2016a, 2016b; Son & Crespo, 2009), this finding shows that a large portion of PSTs in this study developed an insufficient understanding of fraction multiplication. Table 3 shows the frequencies and percentages of the PSTs with correct answers in each problem context. Percentages were rounded to the nearest percent to add up to 100.

Table 2
The frequencies and percentages of PSTs with correct answers in each context

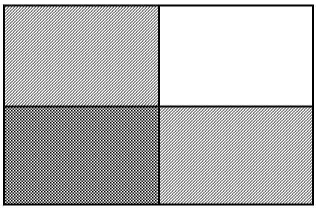
Problem Context	Percent (Frequency)
Word problem (Types 1, 2, 3, 4)	62% (37)
Representation (Types 1, 2, 5, 6)	73% (44)
Computation (Types 1, 3, 5, 7)	66% (40)

Table 2 shows that 62% (37 out of 60 PSTs) recognised the word problem correctly as requiring multiplication of fractions, whereas 73% represented the multiplication of fractions using graphical representations correctly. This finding suggests that our PSTs might have less difficulty solving a multiplication of fractions problem with representation than solving a computational problem or a word problem. Note that this tendency is different from Bezuk and Armstrong's (1995) study in that they reported that their teachers had the most difficulty with representation. In contrast, PSTs in our study seem to show the most difficulty with solving a word problem involving the multiplication of fractions. The following sections describe specific strategies used in each problem context and the types of errors that appeared in each problem context.

PSTs' Performance and Difficulties in Word Problem Contexts

Of 60 participants, 37 PSTs (62%) recognised the given word problem correctly as one in which multiplication of fractions could be applied (types 1, 2, 3, 4). These PSTs explained their solution methods, using graphical representations, symbolic representations, or written descriptions, as shown below.

Example 1 (Graphical representation with written descriptions):



1/3 went to the game by car. I solved this by taking a square to represent the class. I divided it into 4 equal sections and shaded 3. So 3/4 was shaded which was the amount that went to the game. Then this section was already in 3. So 1/3 was shaded. The double shaded area was 1/4 of the original.

Example 2 (Symbolic representation):

$\frac{1}{3} \times \frac{3}{4} = \frac{3}{12}, \frac{1}{4}$ went by car.

Example 3 (Written description)

1/3 of the 3/4 who went to the game went by car. The actual word 'of' tells me to multiply(ing) [*sic*] the fractions to find the amount and then I reduced.

However, around 38% (23 out of 60 PSTs) were not able to translate the word problem correctly as one where the multiplication of fractions could be applied. We identified three types of errors these PSTs made in solving the given word problem involving the multiplication of fractions. First, nine PSTs interpreted the word problem correctly as finding 1/3 of 3/4. However, they did not formulate this into a correct multiplication of fractions equation. For example, instead of formulating $\frac{1}{3} \times \frac{3}{4}$, they often switched the order of fractions as shown below.

$$\frac{3}{4} = \text{football game, } \frac{1}{3} \text{ of } \frac{3}{4} = \text{went by car.}$$

$$\frac{1}{3} \text{ of } \frac{3}{4} = \frac{3}{4} \times \frac{1}{3} = \frac{9}{4}$$

This example shows that when PSTs express 1/3 of 3/4 in words only, we cannot guarantee that they understand the multiplication of fractions and correctly formulate in a numerical format. In fact, there are a lot of PSTs who just answered 1/3 of 3/4 in words in this study. They may ignore convention or language structure.

A second category of errors is a case where PSTs misrepresented the word problem as 3/4 of 1/3. It is also possible that their prior knowledge of commutative property in multiplication prevents them from clearly identifying the unit whole. Among the PSTs who expressed the word problem as 3/4 of 1/3, some formulated it into a correct multiplication expression that presented "3/4 of 1/3" but miscalculated it. Others did not translate it into a correct expression, as in the case of the first category. PSTs with this type of error might not know how to multiply two fractions, so they used cross product multiplication, where the denominator in one fraction is multiplied to the numerator in another fraction. Two examples are shown below.

Example 1: PSTs formulated a fraction multiplication expression correctly, but did not calculate it correctly	**Example 2**: PSTs considered the problem situation involving 3/4 of 1/3 but were not able to formulate a multiplication expression correctly, nor calculate it correctly
$\frac{3}{4} \times \frac{1}{3} = \frac{4}{9}$	$\frac{1}{3} \times \frac{3}{4} = \frac{4}{9}$

A third type of error with the given word problem is the case where PSTs interpreted this problem as a subtraction and/or division of fraction problem (see examples below). The PSTs with this error seemed not to know what operations to use to solve the given problem that involves the multiplication of fractions. In particular, PSTs interpreting a multiplication context as subtraction of fractions may ignore the referent unit, not thinking of a fraction as a relative quantity, comparing 3/4 of the whole and 1/3 of the whole. Some PSTs interpreted it as division (e.g., 3/4 ÷ 3), which may be another example of prior knowledge interference. We noticed that PSTs typically make this type of error when one of the fractional numbers is a unit fraction (i.e., numerator is 1). When there is no unit fraction (for example, in Question 3 where PSTs computed $\frac{3}{4} \times \frac{2}{3} =$), none of the PSTs made such an error. Below are two examples of the third type of error with the given word problem:

$$\frac{3}{4} - \frac{1}{3} = \frac{9}{12} - \frac{4}{12} = \frac{5}{12}$$

I would divide 3/4 by 1/3, which is the same as multiplying 3/4 × 3/1 (the reciprocal). Thus, you get an answer of 3/4 × 3/1 = 9/4 or 2 and 1/4 of the 7^{th} graders went by car. I am using the standard algorithm that I learned for dividing and multiplying fractions ($\frac{3}{4} \div \frac{1}{3} = \frac{3}{4} \times \frac{3}{1} = \frac{9}{4}$).

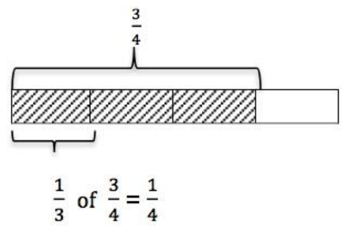
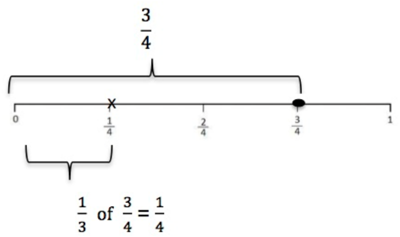
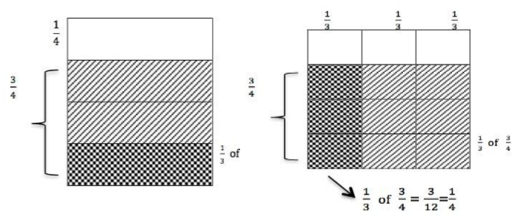
PSTs' Performance and Difficulties in Graphical Representation

About 73% (44 out of 60 PSTs) used graphical representations correctly associated with finding 1/3 of 3/4 (Types 1, 2, 5, 6) (see Table 2). Table 3 shows the percentage of PSTs who used a graphical representation correctly or incorrectly among the participants who solved the word problem correctly. Of the 44 PSTs who answered the word problem correctly, 89% (39 out of 44 PSTs) solved the given word problem with a correct graphical representation. Yet, 11% had difficulty explaining their thinking using graphical representations in solving the given word problem. This finding may show that most of the PSTs who recognised the word problem as a multiplication of fractions tended to be able to explain their thinking using the graphical representations.

Table 3
The numbers and percentages of PSTs' correct responses to Questions 1 and 2

	Total (n=44)
Representation (Types 1 and 2)/ Word problem (Types 1, 2, 3, and 4)	39/44 (89%)
No Representation (Types 3 and 4)/ Word problem (Types 1, 2, 3, and 4)	5/44 (11%)

Figure 2 presents various types of graphical representations used, including fraction bars, area model, number line, part-whole model, and set model. Among the graphical representations, the fraction bars were used most frequently, followed by rectangular area model.

Representation type	Subcategory	Frequency	Example (1/3 of 3/4)
Length model (24)	Fraction bar	23	
	Number line	1	
Area model (18)	Rectangular	14	

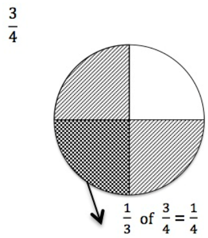
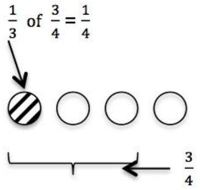
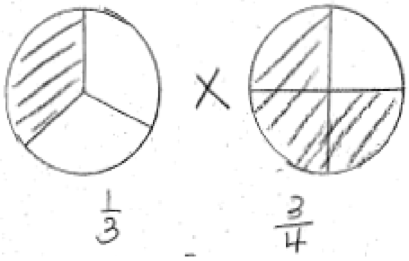
	Circular "pie" pieces	4	
Set model (2)	Set model	2	
Total		44	

Figure 2. The frequencies of graphical representation used with corresponding examples

Sixteen participants failed to use graphical representations correctly in solving the given word problem (Question 2, finding $\frac{1}{3}$ of $\frac{3}{4}$) (Types 3, 4, 7, and 8). The PSTs' difficulties with graphical representations involving "finding $\frac{1}{3}$ of $\frac{3}{4}$" are summarised into three categories. First, some PSTs conceived of partitioning only in relation to a unitary whole; they did not apply partitioning to parts of wholes, and so finding 1/3 of 3/4 was problematic for them. Thus, PSTs in this category drew two presentations—one for 1/3 of a whole and the other for 3/4 of another whole, but did not connect the two representations, as shown below. They are in the *partitioning* stage (Stage 1 discussed above in the Theoretical Background section) because they were not able to apply partitioning to parts of wholes.

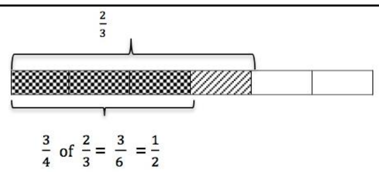
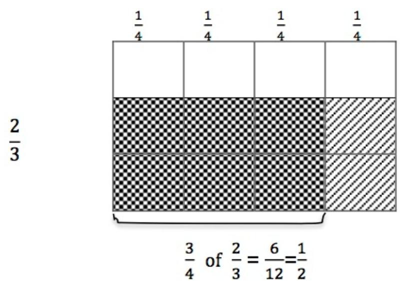


Another type of error relates to PSTs' inability to identify parts of parts (i.e., referent unit). These participants were able to partition parts but did not appear to unite fractional parts into composite units. For example, concerning finding 1/3 of 3/4, the participants were able to partition each fourth into three equal parts, and take one of those three parts. In other words, to find one-third of three-fourths, the students partitioned the given amount into twelve parts total so that they could take three of those parts. However, some PSTs had difficulty solving problems that involved conceiving of fractional parts as composite units. These students could not produce 1/3 of 3/4 as 1/4 or 3/12. They often came up with 3/9, thinking that a whole is three-fourths, failing to conceive of those three parts as three equal parts (twelfths). In taking 1/3 of 3/4, they did not consider finding 1/3 of 3/4 of a whole. So, although these PSTs partitioned parts, they did not appear to capture what Steffe (2003) called *recursive* partitioning, partitioning a partition in service of a non-partitioning goal, such as determining the size of 1/3 of 3/4 of a whole. These PSTs are in Stage 2, *coordination between parts and parts,* where the PST is capable of applying partitioning to parts of wholes and decomposing an initial unit into a unit of units (i.e., recursive partitioning) but is not able to identify a referent unit.

The third category of errors with graphical representations is related to PSTs' insufficient understanding of representing 1/3 of 3/4. They did not know where to start when representing this problem. Rather than first representing 3/4, PSTs' responses in this category represented 1/3 of a whole and then found 3/4 of 1/3.

We also examined the graphical representations used by our PSTs in solving the given computational problem (Question 3, finding $\frac{3}{4}$ ´ $\frac{2}{3}$ =). According to Mack (2000), Question 3 is a next level fraction multiplication problem of Question 2, finding 1/3 of 3/4, which means that Question 3 is more challenging than Question 2 in that the denominator of the first fraction in Question 3 is a multiple of the numerator of the second problem. Indeed, a much smaller number of PSTs correctly used a graphic representation in Question 3. Only 12 of these PSTs are in Stage 3, where the PSTs are able to relate parts of parts back to the original whole. Three models—linear length model (i.e., fraction bar), area model (i.e., rectangle), and set model were used by our PSTs, as shown in Figure 3.

Similar to the problem 1/3 of 3/4, three types of errors were identified in PSTs' graphical representations of finding 3/4 × 2/3. Of the 48 PSTs who failed to use graphical representation correctly, 25 PSTs did not apply partitioning to parts of wholes by presenting 3/4 and 2/3 in a different whole, respectively, or by presenting only one fraction, 2/3 of a whole (Error type 1). Ten PSTs were able to identify 3/4 of 2/3 with a correct graphical representation, as shown in Figure 2, but were not able to represent fractional parts as composite units; they produced 3/4 of 2/3 as 6/8 or 3/4 instead of 6/12 or ½ (Error type 2). We also found error type 2 where PSTs represented 2/3 of ¾ instead of 4/3 of 2/3.

Representation type	Subcategory	Frequency	Example ($\frac{3}{4}$ ´ $\frac{2}{3}$ =)
Length model (4)	Fraction bar	4	
Area model (6)	Rectangular	6	

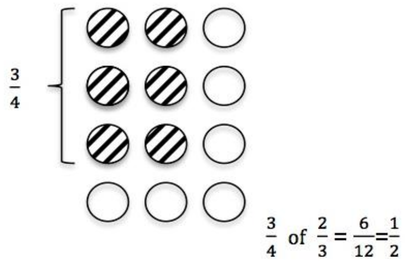
Set model (2)	Counters	2	
Total		12	

Figure 3. The frequencies of graphical representation used with corresponding examples

PSTs' Computation Skill with Fraction Multiplication and Difficulties

Sixty percent (40 out of 60 PSTs) correctly solved the fraction multiplication problem in a purely symbolic notation format, using a traditional standard algorithm (e.g., $\frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{1}{2}$).

Twenty PSTs failed to compute the fraction multiplication problem correctly. Three types of errors were identified, as shown below. The first type of error is a case where PSTs use the addition of fractions in multiplication of fractions problems. Instead of multiplying numerators together and denominators together, they ignored the multiplication symbol and added the two fractions after switching the denominator into a common denominator, as the example below shows.

Written Computation: $\frac{3}{4} \times \frac{2}{3} = \frac{9}{12} + \frac{8}{12} = \frac{17}{12}$

Explanation: First you need to find the common denominator and then multiply the top with the same number with which you multiplied the bottom. Then, you switch the sign over to adding the two fractions.

A second category is the case where PSTs use the multiplication of fractions but do not know how to multiply them. The example below shows that the PSTs multiplied a numerator to a denominator instead of multiplying the numerators and the denominators.

Written computation: $\frac{3}{4} \times \frac{2}{3} = \frac{8}{9} = 1\frac{1}{8}$

Explanation: The first step is to multiply 3 by 3: 3 × 3 = 9 and 4 × 2 = 8. When fractions are multiplied, it is almost like dividing because you are multiplying a small number that is less than 1.

A third error type is the case in which PSTs relied on the common denominator methods while keeping the idea of multiplication. The PSTs in this error found a common denominator and expressed it in multiplication shown below.

Written computation: $\frac{(3 \times 3)}{(4 \times 3)} \times \frac{(2 \times 4)}{(3 \times 4)} = \frac{9}{12} \times \frac{8}{12} = \frac{72}{12}$

Explanation: Two-thirds should be changed into fourths through multiplying by a "common denominator."

PSTs' Identification of Underlying Conceptions of Fraction Multiplication

Two main concepts were identified from the participants: (1) identifying a unit, and (2) finding portions of portions, as highlighted by Steffe and Olive (Olive, 1999; Olive & Steffe, 2002; Steffe, 2002, 2003, 2004). Only 18 out of 60 participants answered the underlying concept of fraction multiplication as finding portions of portions. Two examples of the concepts are shown below, as described by the participants.

- Students need to know how to take a fraction of a fraction and relate that to the whole. It may be necessary to use something similar to fractions of a square to introduce this concept. If solving using the algorithm, they will need to know how to multiply and reduce fractions: 3/4 × 1/3=1/4. They will also need to be able to apply this to a story problem.
- They must know the concept of what a whole is and how to take a part from a whole.

In prior research (e.g., Izsák, 2008; Mack, 2001; Steffe, 2003), researchers recognised that knowledge of partitioning and conceptualising units could be a basis for solving fraction multiplication problems. Similar to these studies, some of the PSTs mentioned students' ability to partition as an essential component for fraction multiplication. They emphasised students' conceptualisation of units and recognising a unit of the units as important concepts to understanding fraction multiplication.

Discussion and Implications

Despite the importance of teachers' mathematical knowledge for teaching and their learning experience in teacher education programs, this study shows that our PSTs have developed insufficient knowledge for teaching multiplication of fractions. In particular, this study shows that many PSTs exhibited varied levels of inflexibility in solving multiplication of fractions problems. Three major blockages identified in this study are as follows: (a) overdependence on mechanical procedures, (b) inflexibility in using multiple modes of representation, and (c) lack of understanding of the concept of unit whole and recursive partitioning process highlighted by Steffe (2003). The following sections include how these blockages presented themselves in PSTs' responses to the tasks used in this study and profiles of the PSTs' mathematical competence on the topic of fraction multiplication by connecting PSTs' ability to solve problems in three contexts.

How do Our PSTs Understand Fraction Multiplication in Three Contexts?

The results of this study showed varied degrees of limited knowledge our PSTs possess in three contexts. While it resonates with prior research addressing teachers' limited conceptual understanding (e.g., Ma, 1999; Son, 2013; Son & Crespo, 2009), we note some differences between this study and other prior research. For example, Bezuk and Armstrong (1995) reported that most teachers examined in their study had difficulty explaining their thinking using representations. In particular, Bezuk and Armstrong reported that there was a lack of language with which to explain their thinking, some confusion in their attempts to draw diagrams to match the algorithmic solutions, and some confusion about the appropriate unit or whole for this problem. However, different from Bezuk and Armstrong's study, our PSTs have little difficulty in explaining their thinking using graphical representations (see Figures 1 and 2). The results of this study show that 73% of the participants could explain their thinking using graphical representations such as a fraction bar and area model. Among the participants who recognised the word problem as a multiplication of fractions, 89% represented the word problem using correct graphical representations. It does not seem surprising that PSTs are better at using representations than they were 20 years ago, given the strong emphasis on that strategy in more recent years. In addition to this finding, this study contributes to the current literature by providing types of difficulties the PSTs had in solving the problems involving fraction multiplication in a word problem format, in a purely symbolic notation format, and in a format requiring the use of visual representation.

For example, in responding to the given word problem, three types of difficulties were identified: (1) some PSTs interpreted the word problem as 1/3 of 3/4, with mistakes in formula or in computation and in the use of graphical representations; (2) some PSTs misinterpreted the word problems as 3/4 of 1/3, also with mistakes in formula or in computation and in the use of graphical representations; and (3) some PSTs misunderstood the word problem as the subtraction and division of fractions involved. These misconceptions may result from (1) some confusion in multiplying two fractions, (2) some confusion about the word "of" in translating the multiplication formula, (3) a misinterpretation of multiplication of fractions, and (4) a lack of PSTs' ability and understanding to coordinate levels of units to presenting "finding parts of parts of a whole." The results of this report may offer possibilities to improve PSTs' understanding of multiplication of fractions in that this study specified possible errors PSTs might make when multiplying fractions.

Furthermore, this study showed that some PSTs did not get the right answer to the word problem using multiplication of fractions even if they could solve the computational problem of multiplication of fractions. They were confused when multiplying two fractions and misinterpreted the word "of" when translating the multiplication formula. This result casts some questions on paper-and-pencil testing, which would present only procedure-oriented problems. This study shows that we could not fully understand student thinking with only one type of problem. Even if students find the right answer to word problems using multiplication and they have the correct computation of the problem on a standardised test, we could not be certain students understand fractional parts in the right way. This study highlights the importance of evaluating PSTs' understanding in various contexts.

What Might be the Origin of PSTs' Errors?

We think our PSTs' misconceptions may result from (1) their lack of understanding about the nature of fractional numbers, (2) too much emphasis on finding a common denominator in fraction addition, (3) their lack of understanding about the meaning of fraction multiplication, and (4) their lack of ability and understanding to partition a partition and decompose an initial unit into a unit of unit structure by relating parts of parts back to the original whole. These hypotheses provide suggestions for teacher educators to consider. For example, when aiding PSTs in developing conceptual understanding of fraction multiplication, teacher educators need to consider every opportunity to help them understand the nature of fractional numbers. Many PSTs have a misconception about the numerator when considering part of a whole. We suggest that the denominators should be addressed as a kind of fractional unit(s) and not by limiting a concept of part-whole. Based on that, they need to emphasise finding a common unit in adding and subtracting fractions. Typically, fraction addition and subtraction are taught procedurally, asking students to find a common denominator and then add the numerators. Teacher educators need to provide a rationale for obtaining a common denominator, referring to the need for a common unit using a real-life example. We cannot simply add 3 cm to 4 mm. In order to find a sum, we need to find a common unit. A similar idea can be used in addressing a common denominator. For example, we cannot add 1/2 and 1/3 because it is one half and one third. Since these are different units, we need to figure out a common unit in order to find the sum. Based on this way of thinking, PSTs could extend the meaning of fractions from repeated addition to finding portions of portions when addressing fraction multiplication. Keep in mind that it is necessary to provide ample experiences in learning how to express an equation and use representations.

Profiles of PSTs' Mathematical Competence in Fraction Multiplication

Solving problems involving fraction multiplication in different contexts requires the following understanding/skills, some of which overlap (e.g., Izsák, 2008; Mack, 2001; Steffe, 2003).

- Ability to partition in relation to a unitary whole
- Ability to demonstrate flexibility in assigning fraction wholes
- Ability to apply partitioning to parts of a whole and identifying parts of part
- Ability to do recursive partitioning
- Understanding of multiplicative conceptual field (multiplicative reasoning)
- Clear understanding of the meaning of numerator, denominator, and the unit whole
- Ability to translate one mode of representation to that of another
- Ability to analyse and understand the mathematical meanings embedded in the various representations
- Ability to identify units and conduct the process of unitising

We saw each level of Steffe's (2003) mental operations: (a) partitioning, (b) coordination between parts and parts, and (c) identification of referent units. Izsák's (2008) descriptions of different levels of cases of employing unit structures in drawn representations exist in our PSTs responses as well: (a) a case primarily reasons with just two levels of units and uses drawings to illustrate solutions, (b) a case more consistently attends to three-level structures and uses drawings to infer a computation method, (c) a case possesses the ability to explicitly produce three levels of units, and (d) a case possesses the ability to explicitly produce three levels of unit structures flexibly so as to compare alternative approaches for producing three-level structures. Based on these prior studies and the results of this study, we propose a *hypothetical* learning trajectory that characterises profiles of the mathematics competence on the topic of fraction multiplication, as shown in Figure 4. We do not argue that the levels in a *hypothetical* learning trajectory must be sequential and that learners typically need to "transition" through each level under instruction. However, there seem to be gradations of competence, ranging from the PSTs who were unable to solve a given problem in any context (Profile 1) to those capable of explaining fraction multiplication in three contexts by flexibly using translations among different representational modes (Profile 5).

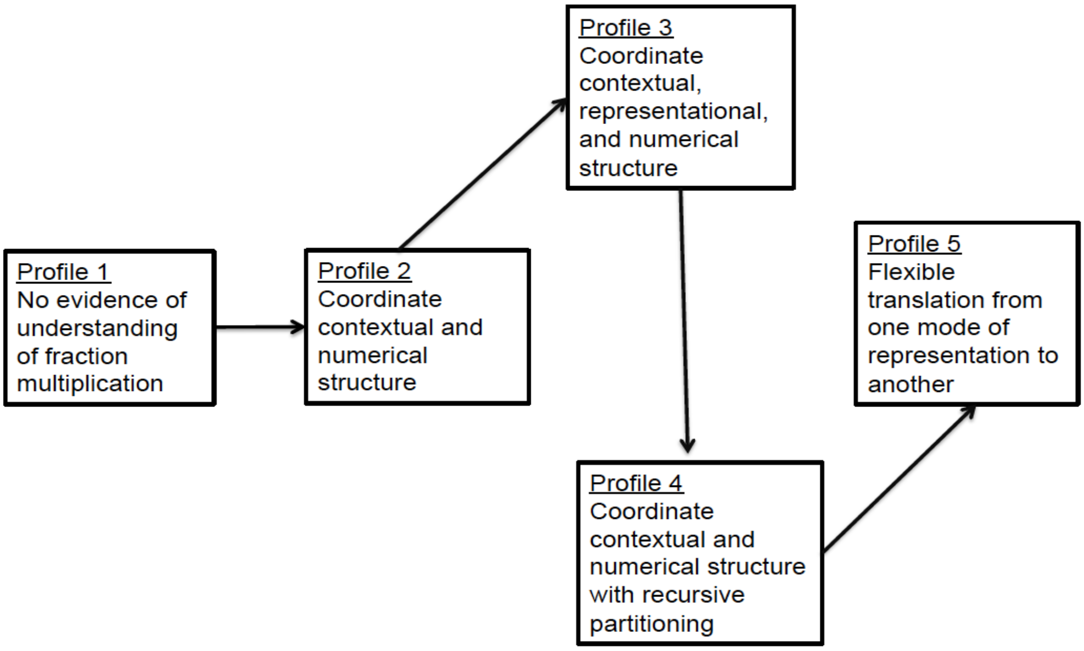


Figure 4. Hypothetical learning trajectories for understanding fraction multiplication in three contexts

We placed the PSTs into five categories as reflected in Figure 4. Of the 60 PSTs, two were able to solve the problems correctly in the three contexts while specifically pointing out the underlying concepts of fraction multiplication (Profile 5); six characterised the understanding from the perspective of a continuum—coordinating contextual and numerical structure with recursive partitioning—that corresponds to the third stage by Steffe (2003) (Figure 1; Profile 4); another six characterised the understanding of the middle area of the learning trajectory (Figure 1; Profile 3) where PSTs were able to translate the word problem context into a correct fraction multiplication sentence, which indicates the second stage by Steffe (2003); 12 PSTs identified only the initial stage of representational knowledge and were able to translate the word problem context into a fraction multiplication sentence but did not provide a correct number sentence (Figure 1; Profile 2); and, finally, 15 PSTs did not show any evidence of understanding the characteristics related to fraction multiplication (Profile 1). More detailed descriptions of each profile are listed below.

Profile 1: No evidence of understanding of fraction multiplication

- Able to understand that the word problem contexts involve fractional numbers
- Not able to translate the fraction multiplication word problem context into a correct number sentence (uses either subtraction or division instead of multiplication)
- Not able to translate the fraction multiplication word problem context into a correct representation

Profile 2: Coordinate contextual and numerical structure

- Able to understand that the word problem contexts involve fractional multiplication

- Able to translate the word problem context into a fraction multiplication sentence but does not provide a correct number sentence (changes the order of fractions or incorrect calculation)
- In using a pictorial representation, able to partition only in relation to a unitary whole and not able to apply partitioning to parts of wholes

Profile 3: Coordinate contextual and numerical structure with recursive partitioning

- Able to understand that the word problem contexts involve fractional multiplication
- Able to translate the word problem context into a correct fraction multiplication sentence
- Calculates correctly
- In using a pictorial representation, able to apply partitioning to parts of wholes and decompose an initial unit into a unit of units but not able to identify a referent unit

Profile 4: Coordinate contextual, representational, and numerical structure

- Able to understand that the word problem contexts involve fractional multiplication
- Able to translate the word problem context into a correct fraction multiplication sentence
- Calculates correctly
- Able to establish three-level unit structures by relating parts of parts back to the original whole
- Not able to identify all the concepts related to fraction multiplication

Profile 5: Flexible translation from one mode of representation to another

- Able to understand that the word problem contexts involve fractional multiplication
- Able to translate the word problem context into a correct fraction multiplication sentence
- Calculates correctly
- Able to establish three-level unit structures by relating parts of parts back to the original whole
- Able to identify all the concepts related to fraction multiplication and analyse and understand the mathematical meanings embedded in the various representations

The proposed transition from one profile to the next is related to the way the PSTs identify relevant mathematical elements related to solving problems in the three contexts, such as coordination between contextual and numerical structure; ability to partition in relation to a unitary whole; ability to apply partitioning to parts of whole; and ability to identify parts of part (referent unit) and do recursive partitioning correctly. We put forward the gradation and transition from one profile to another (Figure 4). The step from Profile 1 to Profile 2 involves the PSTs being capable of using the mathematical element "coordination of contextual, numerical, and representational structures" to solve the problem. The step between Profiles 2 - 3 and 4 takes place when PSTs are able to recognise the relevance of "recursive partitioning" as a mathematical element to solve problems, since the PSTs in Profiles 2 and 3 do not clearly demonstrate this understanding. The progression from Profiles 3 and 4 to Profile 5 takes place when PSTs are capable of recognising an element relevant to fraction multiplication, based on the identification of the referent unit. Most PSTs who recognised the word problem as a multiplication of fractions were able to explain their thinking using the graphical representations. These findings offer descriptors of how the PSTs understand fraction multiplication in different contexts and provide information for the design of intervention in teacher education, one objective of which would be to support recognition of the connectivity of fraction multiplication.

Implications, Limitations and Future Directions

The findings of this study contribute to our understanding of how PSTs understand the topic of fraction multiplication. However, our findings are based on a survey of a relatively small group of PSTs. Thus, the findings should not be overgeneralised or used as criteria for determining the quality of teacher education programs.

Although many studies investigated children's understanding of fractions, a relatively small number of studies has been done in teacher education programs. Studies on PSTs are generally limited to simply addressing their insufficient knowledge. In this regard, our findings provide more specific information for the design of teacher education materials that take into account the characteristics of the PSTs' learning. In particular, the details of the profiles identified provide information for describing the progress of the PSTs in their development of mathematical competence within the context of fraction multiplication.

The evidence concerning PSTs' understanding and errors made in different contexts sheds some light on the essence of teacher knowledge. Different errors involving fraction multiplication have implications for teacher educators. This study has implications for teacher educators working to design mathematics courses for PSTs, as well as for researchers interested in better understanding teacher knowledge and teaching strategies. The findings of this study suggest teacher education curriculum needs to be more problem-based, where PSTs have opportunity to solve problems in different contexts are asked to explore the transition of one mode of representation to that of another, and are asked to interpret and respond to student thinking in various contexts. Teachers must conquer their own obstacles with mathematical knowledge before being able to help their students (McClain, 2003; Son, 2016a, 2016b; Son & Crespo, 2009; Son & Sinclaire, 2010). To this end, teacher educators may need to re-organise the repertoire of teacher education courses or refine the instructional sequence. Our profiles suggest that "coordination of contextual, numerical, and representational structures," "recursive partitioning," and "identification of the referent unit in the nested level structure" are critical indicators of understanding fraction multiplication. Future research will explore what types of approaches or opportunities can be provided in order to address these indicators for those PSTs who demonstrate understanding at different levels of profiles. It is our hope that understanding the profiles reported in this study will serve as a helpful framework for future studies.

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