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Diagnostic Assessment of Pre-Service Teachers' Mathematical Content Knowledge

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Establishing the mathematical content knowledge of pre-service teachers is a requirement in New Zealand. However, this knowledge is not clearly defined and is a challenge for initial teacher education providers to assess. We develop the concept of foundation content knowledge and suggest how it can be assessed using two forms of diagnostic assessment. We examined an established conventional written test and an adaptive online adult numeracy assessment. The usefulness of each assessment tool was analysed in terms of how it revealed the foundation content knowledge of pre-service teachers. The two assessment tools were administered to different cohorts of pre-service teachers. In both cases, less than one-half of each cohort demonstrated foundation content knowledge at the beginning of their teacher education programmes. Both assessment tools provide information about the learning needs of each pre-service teacher. Crucially, the adaptive assessment tool gave immediate feedback directly to pre-service teachers. Furthermore, the adaptive assessment tool has the potential to provide consistent diagnostic information from year to year and between initial teacher education providers.

Introduction

Mathematical Knowledge for Teaching

> No one questions the idea that what a teacher knows is one of the most important influences on what is done in classrooms and ultimately what students learn. However, there is no consensus on what crucial knowledge is necessary to ensure that students learn mathematics. (Fennema & Franke, 1992, p. 145)

The mathematics education research community has documented a relationship between the mathematical knowledge of teachers and students' achievement (e.g., Bobis, Higgins, Cavanagh, & Roche, 2012; Hill, Ball, & Schilling, 2008; Rowland & Ruthven, 2011; Senk, Tatto, Reckase, Rowley, Peck, & Bankov, 2012). What remains contested, as expressed by Fennema and Franke above, is what this knowledge consists of and how it is used by teachers with their students. Mathematics researchers have built upon Shulman's (1986, 1987) ideas to develop their own models of mathematical knowledge for teaching (Rowland & Ruthven, 2011). In particular, Shulman (1986) reminded us about the important relationship between knowledge and teaching when he identified content knowledge as significant dimension of teaching. Similarly, Fennema and Franke (1992) have developed a contextualised model that emphasises the dynamic interaction between four different types of knowledge used by teachers. In contrast to this contextual approach, Delaney, Ball, Hill, Schilling, and Zopf (2008) have developed a practice-based theory that consists of two domains: subject matter knowledge and pedagogical content knowledge.

Common to these three models of teachers' mathematical knowledge is the dynamic function of mathematics knowledge, because it is required to interact with the other forms of teaching knowledge and be transformed by a teacher into practice. However, descriptions of mathematics content knowledge evident in the literature (e.g., mathematical horizon knowledge, Ball & Bass, 2009; contextually bound features, Fennema & Franke, 1992; profound understanding of fundamental mathematics, Ma, 1999) contain features of teaching that have been identified from and associated with expert teachers. Pre-service teachers may not have encountered these aspects of mathematics content knowledge. This knowledge will be an aspiration for most primary pre-service teachers at the beginning of their initial teacher education programmes.

Mathematical Content Knowledge

As we considered the salient aspects of mathematical content knowledge in relation to pre-service teachers, we returned to Shulman's ideas:

> To think properly about content knowledge requires going beyond knowledge of the facts or concepts of a domain. It requires understanding the structures of the subject matter [where] the substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts. The syntactic structure of a discipline is the set of ways in which truth or falsehood, validity or invalidity, are established. (Shulman, 1986, p. 9)

Shulman's inclusion of both substantive and syntactic knowledge structures resonates with many cognitive descriptions of content knowledge (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986; Ma, 1999; Porter, 2002; Skemp, 1976). For example, Hiebert and Carpenter describe conceptual knowledge of mathematics as a web-like representation where prominent pieces of information are interconnected, whereas procedural knowledge of mathematics is described as the symbolic representation system and algorithms for completing mathematical tasks. When procedural knowledge representations have limited connections between successive actions, then they become efficient to execute because they require minimal cognitive effort. Both are needed for solving problems effectively, and their interdependency becomes particularly prominent when the problems require interpretation to arrive at a reasonable solution (Holmes, 2012). We should not, however, interpret conceptual and procedural knowledge as alternative conceptualisations of the nature of mathematics, but rather as intertwined manifestations of mathematical knowledge.

Further developing this dialogic line of reasoning about mathematical knowledge, Mason, Stephens, and Watson (2009) suggest that although mastering procedures is important in mathematics, remembering them and using them in isolation from conceptual knowledge places too much load on memory. Instead, they view mathematical content knowledge as an amalgam of formal mathematical concepts and procedures. They term this amalgam an "appreciation of mathematical structure" (p. 12) where the relationships between mathematical ideas can be used as criteria for determining the appropriateness of a procedure to solve a particular problem. For procedural knowledge to be used effectively, it has to be appreciated in relation to the structural features of the formal mathematical content and problem context in which it is to be used.

Mason et al.'s (2009) emphasis on the importance of the contextual aspects of mathematical knowledge echoes the earlier work of Porter (2002), who examined the level of cognitive demand of mathematical problems. Porter described these levels in terms of cognitive complexity from simple to complex: memorise; perform procedures; communicate understanding; solve non-routine problems; and conjecture, generalise, or prove. Accordingly, problem solving requires integrating conceptual and procedural knowledge. For the purpose of our study, Porter's integrated model provides a succinct description of the aspects of mathematical content knowledge relevant to our investigation.

Mathematical Content Knowledge Expected of Pre-service Teachers

Not only has the assessment of mathematics content knowledge of qualified teachers become an area of international interest (Stephens, 2003), but assessing the mathematical content knowledge of pre-service teachers is also a growing focus (Senk, et al., 2012). Although the New Zealand Teachers Council requires initial teacher education providers to assess candidates' numeracy competency prior to entry and to ensure that pre-service teachers meet the requirements prior to graduation (New Zealand Teachers Council, 2010), it does not explicitly specify the numeracy requirements. Instead, the Council rather vaguely requires that initial teacher education providers demonstrate how graduates have met the graduating teaching standards. These standards require graduates to have the content knowledge appropriate to the learners and learning areas of their programme.

In the case of primary pre-service teachers, they should have mastery of mathematics content knowledge to at least Level 4 of the New Zealand Curriculum (Ministry of Education, 2007). The mathematics topics at Level 4 are similar to the Year 7 content descriptions of the Australian Curriculum (Australian Curriculum Assessment and Reporting Authority, 2012) and Grade 6 descriptors of the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010). As in Australia and the United States, the topic of number constitutes a large part of the primary school mathematics curricula, and therefore teacher knowledge of number concepts and operations is particularly important. At Level 4 of the New Zealand Curriculum, there is an emphasis on using and operating on fractions, decimals, and percentages, and extending understanding of place value to decimals. Additionally, the New Zealand Curriculum prefaces each mathematics achievement objective with the phrase, "In a range of meaningful contexts, students will be engaged in thinking mathematically and statistically. They will solve problems and model situations that require them to ..." (Ministry of Education, 2007, p51). The achievement objectives also place an emphasis on understanding, generalisation, and using a range of representations. This emphasis is consistent with the integrated view of mathematical knowledge discussed in the preceding section and is consonant with curriculum developments in Australian and the United States.

In a major international study the content knowledge of 15,163 pre-service teachers from 451 institutions in 16 countries was investigated (Senk, et al., 2012). Rasch analysis (Wright & Masters, 1982) was used to estimate item difficulty and person ability and hence provide qualitative descriptions of the mathematical content knowledge of clusters of pre-service teachers. Senk and colleagues found that only 50% of primary (to grade 6) pre-service teachers in the United States were likely to be able to use fractions to solve story problems, recognize examples of rational and irrational numbers, know how to find the least common multiple of two numbers in a familiar context, recognise that some arguments about whole numbers are logically weak, and determine areas and perimeters of simple figures. Also the same group of pre-service teachers were likely to have trouble reasoning about factors, multiples, and percentages, even though they were likely to solve some problems involving proportional reasoning. We view these descriptive benchmarks as useful indicators for assessing the mathematics content knowledge of pre-service teachers.

Ryan and McCrae (2005/6) identified pre-service teachers' misconceptions in the topics of number and measurement concerning place value, decimal computation, fractions, fraction computation, calculator displays, ratio, conversions of units, and area and perimeter. They also found misconceptions in the topics of probability, statistics, geometry, and algebra. Similarly Tirosh and Graeber (1989) found that many pre-service teachers at a United States university held misconceptions about multiplication and division and had only a procedural understanding of division by decimals. Afamasaga-Fuata'i, Meyer, Falo, and Sufia (2007) found that no Samoan pre-service teachers displayed 80% or more mastery of content questions derived from the TIMSS 1999 study for 15 year olds; their common difficulties included ordering of fractions and decimals, calculating percentages, addition of fractions, multiplication of decimals, converting units for computed rates, and interpreting word problems. Similarly, in a study of two rural Australian university campuses Tobias and Itter (2007) found that only 10% of beginning pre-service primary teachers displayed 75% or more mastery of questions focussing on place value, fraction and decimal understandings, along with problem solving skills appropriate for lower secondary students. In New Zealand, Ward and Thomas (2007) demonstrated that it is not only pre-service teachers who have difficulties in the area of rational number. They found that approximately one-third of practising teachers in New Zealand also display a lack of proficiency in the addition and division of fractions and proportional reasoning. The literature paints a disturbing picture of a large group of pre-service teachers and teachers with incomplete mathematical knowledge.

Foundation Content Knowledge

The area of pre-service teachers' mathematical content knowledge has been theorised predominantly from a deficit perspective (Ryan & Williams, 2011). For example, the mathematics content knowledge of a significant proportion of pre-service primary teachers entering university has been portrayed as a lack of proficiency (Anthony, Beswick, & Ell, 2012; Hamlett, 2010). If mathematics tests used to assess pre-service teachers' mathematical content knowledge consist of isolated procedural knowledge and methods of solutions, then they do not provide information as to whether or not pre-service teachers demonstrate integrated understanding of mathematics content knowledge. This limited snapshot of pre-service teachers' mathematical content knowledge has been noted by contemporary mathematics education researchers (Ball, 2000; Fennema & Franke, 1992; Hill, Schilling, & Ball, 2004; National Mathematics Advisory Panel, 2008) and efforts have been made to examine pre-service teachers' mathematical content knowledge from a growth-oriented perspective. Ryan and Williams (2007) have demonstrated this growth-oriented view by involving pre-service teachers with the interpretation of their own diagnostic assessments of mathematical content knowledge. Like Ryan and her colleagues, we recognise pre-service teachers' potential to interact with their own mathematical content knowledge with greater agency. Hence we use the term foundation content knowledge to refer to the mathematical content knowledge that pre-service teachers bring with them and are able to demonstrate at the beginning of their initial teacher education programmes.

Currently, the literature describes the absence of mathematical content knowledge in terms of procedural errors (i.e., mistakes in computation), misconceptions (i.e., conceptual flaws), or incorrect strategies (i.e., inappropriate application of a concept or procedure). The conventions used to discuss responses that are incorrect from a formal view of mathematics are reflective of a firmly established deficit view of mathematics knowledge. At this time, however, we too are bound by the deficit language available in the literature to illustrate how we assessed the concept of foundation content knowledge.

Since the errors and misconceptions commonly held by school students are also prevalent in pre-service teachers (Ryan & McCrae, 2005/6) we have used knowledge of these errors and misconceptions as a guide for writing the conventional test questions to assess foundation content knowledge. The "Concepts in Secondary Mathematics and Science" study (Hart, 1981) is a valuable source of information on school student performance, errors, and misconceptions, as is the contemporary "SMART Assessment for Learning" project (Stacey, Price, Steinle, Chick, & Gvozdenko, 2009). Other reports of tests to assess pre-service teachers' content knowledge (e.g., Afamasaga-Fuata'i et al., 2007; Tobias & Itter, 2007), typically make use of national curriculum documents appropriate to the level at which pre-service teachers will be expected to teach, questions from international studies of school students, and errors and misconceptions identified in the research literature. These resources have provided us with a bank of information from which we can make corroborated inferences about pre-service teachers' foundation content knowledge.

Adaptive Assessment in New Zealand

We are interested in efficiently assessing the foundation content knowledge of pre-service teachers and, like Ryan and her colleagues, assisting pre-service teachers to better understand their own mathematical content knowledge. Unfortunately, at present there is no common assessment tool used by initial teacher education providers in New Zealand to measure numeracy competency, foundation content knowledge, or the mathematical content knowledge appropriate for teaching, as required by the New Zealand Teachers Council. To the best of our knowledge, the assessment tool described by Ryan and McCrae (2005, 2005/6) is not being used by any initial teacher education providers in New Zealand. The Tertiary Education Commission (the body responsible for funding tertiary education in New Zealand) has recently developed an adaptive online adult numeracy assessment tool (Tertiary Education Commission, 2010). This adaptive assessment tool was designed for use by tertiary education providers to assess students in their vocational courses and focuses on content knowledge of number and measurement. We investigated whether this adaptive assessment tool might be useful for measuring pre-service teachers' foundation content knowledge.

Methods

The research was carried out over two years at the University of Otago College of Education. Pre-service teachers' performance in a conventional test of foundation content knowledge and an adaptive assessment with a focus on foundation content knowledge of number and measurement was examined. The research questions addressed were:

- Do pre-service teachers display the foundation content knowledge considered necessary for teaching?
- Can an adaptive online assessment tool be used to assess whether pre-service teachers are in need of support?

Participants

Participants for this study were pre-service teachers in their first year of study in the Bachelor of Teaching and Bachelor of Educational Studies programmes at the University of Otago College of Education. In 2010, 153 of 154 pre-service teachers consented for their data to be used for research purposes, and in 2011 all 122 pre-service teachers consented.

Assessment Tools

Conventional Test. In 2010, the foundation content knowledge of the pre-service teachers was assessed by a conventional test. This test asked questions related to the broad mathematics curriculum up to the Year 9 level of schooling in New Zealand, and used a format similar to tests used in previous years. In particular, the foundation content knowledge was situated in meaningful contexts that required pre-service teachers to use more than just procedural knowledge to answer the questions correctly. Pre-service teachers were given access to a parallel practice test three weeks prior to the assessment in March. The test consisted of five sections: 20 questions on number, 10 on algebra, 12 on measurement, 6 on geometry, and 12 on statistics and probability. To pass this test, pre-service teachers needed to respond correctly to 45 of 60 questions (i.e. 75%).

As our objective was to assess whether pre-service teachers displayed both the substantive and syntactic aspects of foundation content knowledge appropriate to primary education, the test contained a number of carefully considered features. First, most of the questions were placed in contexts likely to be meaningful to the pre-service teachers. Question features such as "how much do they cost now?" were used to ensure that pre-service teachers displayed a conceptual understanding of mathematics by applying their knowledge and strategies in a contextual problem solving situation. Second, pre-service teachers were not permitted to use calculators. All questions were designed so that answers could be computed using elementary numeracy strategies. In general, answers to most questions were designed to be whole numbers, except for those specifically addressing fraction and decimal knowledge. This question feature ensured that pre-service teachers displayed sound knowledge of basic facts and could not hide their lack of knowledge behind technology (e.g., by using fraction keys on calculators). Third, questions were chosen so that they would reveal common errors and misconceptions, such as when adding fractions (Hart, 1981) and ordering decimals (Stacey & Steinle, 1998).

Adaptive assessment tool. In 2011, the foundation content knowledge of pre-service teachers was assessed by the Tertiary Education Commission's (TEC) literacy and numeracy for adults online assessment tool (Tertiary Education Commission, 2010). The pre-service teachers were supervised in a computer laboratory while they sat this test. No time limit was set, but most of them completed the assessment in less than one hour. Each pre-service teacher was asked 30 questions on general numeracy drawn in an adaptive manner from a large bank of questions on number knowledge, number strategies, and measurement. The numbers of available items at each numeracy step are shown in Table 1. Our threshold score required pre-service teachers to achieve Step 6, which included requirements for them to solve addition and subtraction problems involving fractions using partitioning strategies; solve multiplication or division problems with decimals, fractions, and percentages using partitioning strategies; use multiplication and division strategies to solve problems that involve proportions, ratios, and rates; and know the sequences of integers, fractions, decimals, and percentages, forwards and backwards, from any given number. Full details of the progressions of requirements at each step are given on the TEC website http://literacyandnumeracyforadults.com/The-Learning-Progressions.

The assessment tool adapts to whether a user is getting questions right or wrong. The software draws upon its large bank of questions and continuously adjusts the difficulty of questions presented to the user, until the user is responding correctly to questions at that level of numeracy half of the time. In this way it presents questions that are neither too hard nor too easy to be useful for diagnostic purposes. Therefore, each pre-service teacher was assessed with a different selection of questions to other pre-service teachers. This tailoring of the adaptive assessment to responses of each pre-service teacher made invigilation relatively easy, because most of the time everyone was working on a different question. Like the 2010 study cohort, the 2011 pre-service teachers were given access to a practice assessment three weeks prior to the assessment to prepare for the adaptive assessment; however, this practice assessment was in written form and therefore not adaptive.

Inspection of the Step 5 and Step 6 questions in the adaptive assessment suggested they were equivalent in terms of content, level of difficulty, and contexts to the questions in the number and measurement sections of the conventional test used in 2010.

Table 1
Numeracy Questions by Step Level

Step	Number of items
1	111
2	120
3	229
4	228
5	276
6	165

All questions in the adaptive assessment were presented in contexts likely to be meaningful to the pre-service teachers and required them to apply their knowledge and strategies to solving problems. Each question had a visual prompt that reinforced the context, as in the example in Figure 1 on the following page. The adaptive assessment questions varied in response format: Students were required to either type in a numerical answer, select from a number of choices presented on screen, agree or disagree to offered statements, or rearrange sequences using a drag and drop facility.

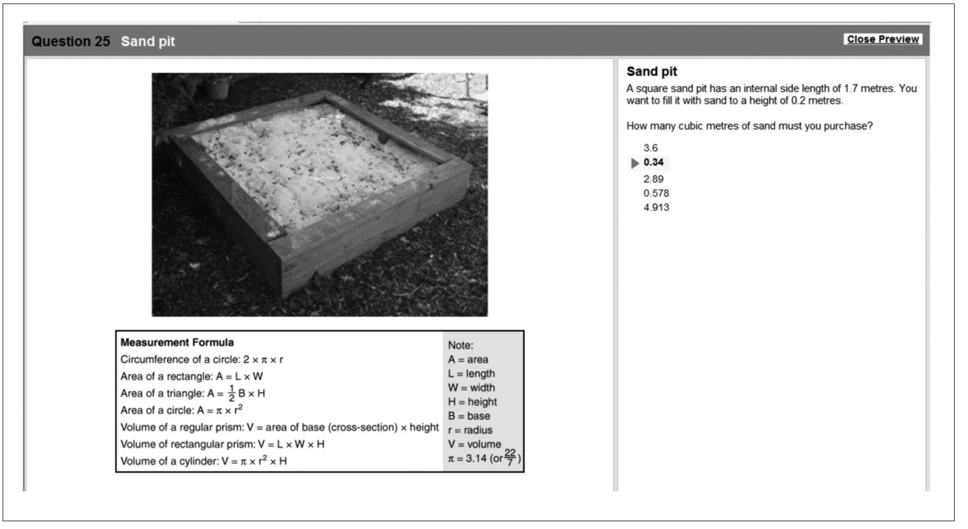

Figure 1. Example of adaptive assessment question

Pre-service teachers were permitted to use simple non-scientific calculators to compute their answers for some questions. When no calculators were permitted for a particular question, then a "no calculator'" sign was displayed on the screen.

The adaptive assessment tool presented pre-service teachers with their results on screen immediately after they completed the assessment. They were given their score and a clear indication of whether they had been assessed as having sufficient foundation content knowledge or not. They were also informed about which questions they had answered correctly and incorrectly. This information provided them with the opportunity to better understand and develop their mathematics content knowledge. In addition, through Rasch modelling, the adaptive assessment tool provided an estimate of the uncertainty of each pre-service teacher's score, which enabled teacher educators to make informed decisions about the results.

Results

Overall results

Conventional test. Pre-service teachers were required to score 75% in order to pass the conventional test. In 2010, only 62 pre-service teachers (41% of the cohort) passed the test and the distribution of their scores is shown in Figure 2.

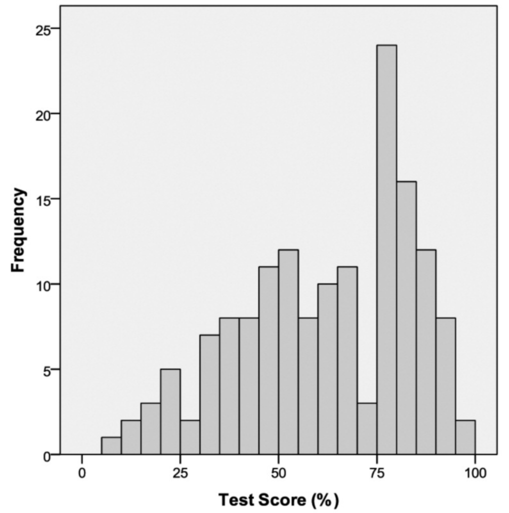

Figure 2. Distribution of marks from conventional written test given to pre-service teachers in 2010

Of note is the smaller number of pre-service teachers scoring between 70% and 75% as compared to the number scoring between 75% and 80%. This variation is the result of lenient marking. The tests of pre-service teachers that were within 5% of a passing score were re-examined and re-scored. The rationale for re-scoring tests that were within 5% of the pass mark was the fact that it would be unfair to penalise a pre-service teacher if they demonstrated appropriate conceptual knowledge but made minor procedural errors. Records of conventional test results from previous years indicated that the proportion of pre-service teachers who passed in 2010 was similar to prior cohorts. This suggests that the difficulty of the test and the foundation knowledge of pre-service teachers may be stable from year to year.

Adaptive assessment. The distribution of scores for the adaptive assessment is shown in Figure 3. We decided, before any students sat the assessment, to make the pass mark at the break point between Step 5 and Step 6. This decision was based upon two considerations. Firstly, the descriptors of Step 6 provided by the Tertiary Education Commission were broadly equivalent to the numeracy component of foundation content knowledge we had specified for our pre-service teachers. Secondly, we completed the assessment a large number of times, using a different specified random error rate each time and adopting specific (mis)conceptions that are common among students and pre-service teachers (Hart, 1981; Stacey at al., 2009; Ryan & McCrae, 2005/6; Senk et al., 2012). These simulations also suggested to us that Step 6 was a suitable pass mark. The adaptive assessment was given to pre-service teachers in 2011. Only 52 pre-service teachers (43% of the cohort) achieved a score of Step 6 and we noted that this result was similar to the pre-service teachers' performance in the conventional test from 2010.

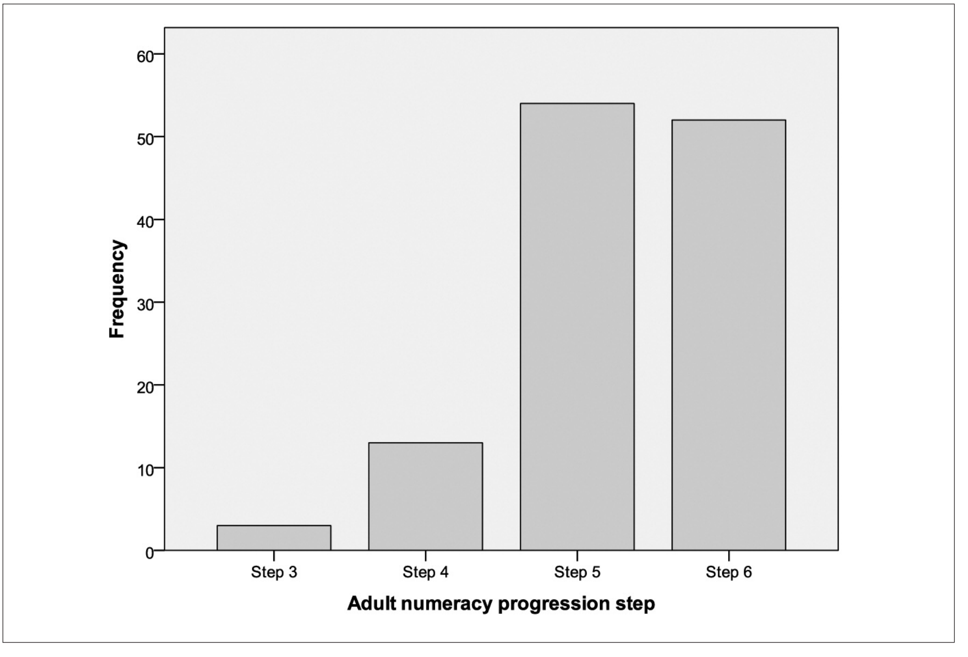

Figure 3. Distribution of pre-service teachers' scores in adaptive assessment

Comparing the conventional and adaptive assessments

Because the adaptive assessment tool presents each pre-service teacher with a different set of questions, we encountered significant challenges when we attempted to compare the results from this assessment with the conventional test. Very often, only about five pre-service teachers were presented with any particular question, so we could not simply compare the number of pre-service teachers answering each type of question correctly. However, we identified a selection of questions from each assessment that were assessing parallel foundation content knowledge, though the particular contexts in which the knowledge was presented differed. Significantly, all of the questions selected from the adaptive assessment were classified at Step 5 or Step 6 by the designers of the tool. We decided to examine these matching questions to determine if these two assessments could be compared meaningfully.

We identified a set of questions from the conventional test that could be used to indicate foundation content knowledge. We termed these "indicator questions" because they provided us with sufficient and representative evidence to establish whether or not a foundation content knowledge was demonstrated instead of considering every question in the test. Twelve indicator questions were chosen for this study on the basis of four considerations: correspondence with descriptions of the performance of pre-service teachers in the study by Senk et al. (2012), the importance of number and measurement in New Zealand primary schools, our own prior experience of pre-service teachers' difficulties with these types of questions, and our observations of the pre-service teachers' difficulties teaching mathematics topics to children.

The 12 indicator questions from the conventional test selected were: placing the fraction 74 on a number line; multiplying four thousand by two hundred thousand; adding one-eighth and two-fifths; ordering decimals; reducing $120 by 25%; dividing $132 by 3; finding the difference between 14.96 and 19.8; ordering the fractions 23, 34 , 35 from smallest to largest; expressing 16 out of 20 as a percentage; multiplying the fractions 34×14; calculating the volume of a cuboid; converting 8,000cm3 to litres; and finding the area of a simple compound L shape. All these indicator questions were in contexts likely to be meaningful to pre-service teachers except for the questions on placing fractions on a number line and ordering fractions, which were context-free. The distributions of pre-service teachers' scores for all twelve indicator questions for those who passed and those who failed the conventional test are shown in Figure 4.

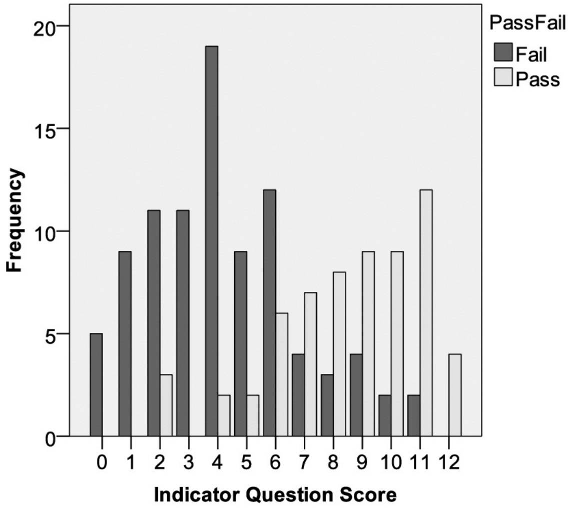

Figure 4. Distribution of pre-service teachers' scores for indicator questions

As the twelve indicator questions constituted one fifth of the total questions in the test, it is not surprising to see a strong correspondence between the indicator question score and passing the test. However, it is noteworthy that of the 153 pre-service teachers completing the conventional test, only eight of them with an indicator score of 9 or more failed the test. This finding implies that a high indicator question score is a reasonable indicator of foundation content knowledge. The converse is not true, however, because 28 pre-service teachers who had an indicator question score of less than 9 also passed the conventional test. Even though these 28 pre-service teachers did not demonstrate foundation content knowledge by answering nine or more indicator questions correctly, they answered over 75% of the other questions in the test correctly. These other questions, derived from level 4 of the New Zealand Curriculum, may have been easier than the questions we nominated as indicator questions. This suggests that it is possible to pass the conventional test without foundation content knowledge, raising questions about how valuable the conventional test is for diagnostic assessment of pre-service teachers.

Matched question analysis

Once we established that the twelve indicator questions from the conventional test were appropriate indicators of foundation content knowledge, we looked for equivalent questions in the adaptive assessment data. In some cases, there was only one question from the adaptive assessment that was equivalent to the conventional test, whereas in other cases, there were a number of equivalent options for us to choose from. For example, the question that required pre-service teachers to calculate the percentage of an amount in the conventional test had at least 20 parallel questions in the adaptive assessment. In the conventional test, this question was situated in the context of a 25% discount on $120 back-country hut passes. In the adaptive assessment, among the 20 identified parallel questions were contexts including finding: a 30% discount for $90 toasters, 40% of 90 squares in a patchwork, a 10% discount for a $48 pizza bill, a 20% off special offer on a $320 membership fee, and identifying the correct method for finding 15% of $120. Interestingly, by using Rasch analysis the Tertiary Education Commission had placed some of these questions at Step 5 and some at Step 6, emphasising just how critical the context, wording, and particular values are for creating item difficulty.

Due to space limitations, we report the responses pre-service teachers gave to only four of the 12 matched questions. The matched questions are presented as pairs and each indicator question from the conventional test is followed by its matched parallel question from the adaptive assessment. Indicator questions are described as they appeared in the conventional test. Common categories of pre-service teacher responses are listed from most to least frequent except for the categories of “other” and “no response”. “Other” responses include those that did not match a conception reported in the literature or occurred with a frequency of less than 3%. In keeping with the growth orientation discussed in the introduction, inferences about the foundation knowledge demonstrated by pre-service teachers are discussed as conceptions rather than misconceptions or "errors". Instead, errors refer to specific responses that are incorrect in relation to the formal mathematical concepts.

Along with each indicator question, we present a parallel question selected from the adaptive assessment data. Categories of responses were determined by the designers of the adaptive assessment tool. In the four examples we have selected to report below, adaptive assessment users could only choose from four or five possible responses rather than generate their own responses as in the conventional test. For the parallel questions that permitted calculator use, we considered these to be equivalent to the less computationally demanding questions in the conventional test, for which calculators were not allowed. Frequencies are not reported for the parallel questions because of the small numbers of pre-service teachers who were presented with each question.

1. Matched questions about place value. Table 2 shows pre-service teachers' responses to an indicator question intended to reveal foundation content knowledge of place value. Only 39% of pre-service teachers gave a response considered correct. Sixteen of the correct responses were written in non-standard form. Specifically, eight had idiosyncratic spacing and/or comma placement (e.g., 80,0,00,0,000) and eight had no spacing or commas between digits (i.e., 800000000). Only one correct response and eight incorrect responses were written in words rather than symbolic form. Thirty-seven percent of pre-service teachers gave incorrect responses of 800,000 or 8,000,000. These responses were associated with specific conceptions shown in Table 2.

Table 2
Calculate four thousand multiplied by two hundred thousand

Response	Frequency	Inferred Conception
800,000,000 or 800 000 000	39%	Foundation knowledge likely
800,000 or 800 000	25%	Computing 4 times 200 and considering the units to be thousands
8,000,000 or 8 000 000	12%	Procedural error in multiplying powers of ten
Other	21%	Not interpreted
No answer	3%	Uncertain how to respond

The question shown in Table 3 is an example of an adaptive assessment item that we identified as equivalent to the indicator question reported in Table 2. This parallel question was designed to find whether pre-service teachers could multiply a given whole number by 10,000. Specifically, the parallel question was worded "Multiplying by 1000: 304 × 10,000 equals" and calculators were not permitted. The parallel question was at Step 5 of the adult numeracy framework and probably had an item difficulty less than the indicator question because a lower place value was involved. From these findings, we infer that at least 39% of the 2010 cohort was performing above Step 5 on the adult numeracy framework for the topic of place value.

Table 3
Multiply 304 × 10,000

Response	Inferred Conception
3,040	Multiply by 10; or value is a number in the 1,000s; or possibly reading the comma as a decimal point
30,400	Multiply the first multiplicand by 100; or the product is expressed as the value of the first multiplicand in the 10,000s
304,000	Multiply the first multiplicand by 1,000; or the product is expressed as the value of the first multiplicand in the 100,000s
3,040,000	Foundation content knowledge likely
3,400,000	Counting zeros in both multiplicands and adding them to the product of 34 × 1

2. Matched questions about the addition of fractions. Table 4 shows pre-service teachers' responses to an indicator question intended to reveal foundation knowledge of adding fractions with different denominators. Specifically, the indicator question was worded, "On a previous tramp one eighth of the people I met were Israelis, two fifths were European and the rest were Kiwis. What fraction of the trampers were from overseas?" Only 37% of pre-service teachers gave a correct response by converting each fraction to a common denominator. The conception of treating the numerators and denominators as separate whole numbers is prevalent in school students, with approximately 20% of 12 to 15 year olds making similar errors (Hart, 1981).

The question shown in Table 5 is an example of an adaptive assessment item that we identified as equivalent to the indicator question reported in Table 4. This parallel question was designed to find whether pre-service teachers could add two common fractions less than 1. Specifically, the parallel question was worded, "An earthquake destroyed 14 of the houses in a town. Another 35 of the houses were badly damaged. What is the total fraction of the houses either destroyed or badly damaged?" The parallel question was at Step 6 of the adult numeracy framework and probably had an item difficulty approximately the same as the indicator question. From these findings, we infer that approximately 37% of the 2010 cohort was performing at Step 6 on the adult numeracy framework for the topic of addition of fractions with different denominators.

Table 4
Adding one-eighth and two-fifths

Response	Frequency	Inferred Conception
2140	37%	Foundation knowledge likely
313	12%	Treating numerators and denominators as separate whole numbers
Other with 40 as denominator	5%	Procedural error after converting fractions to a common denominator
1940	3%	Interpretation error of context
Other	26%	Not interpreted
No answer	16%	Uncertain how to respond

3. Matched questions about ordering decimals. Table 6 shows pre-service teachers' responses to an indicator question intended to reveal foundation knowledge of ordering decimals. Specifically, the indicator question was worded, "These are the exact weights of four trampers' packs: Ali, 19.098kg; Ben, 19.2kg; Carl, 19.08kg; Dion, 19.2005kg. Write these weights in order from lightest to heaviest." Seventy-three percent of pre-service teachers gave a correct response, whereas 15% of pre-service teachers gave responses indicating conceptions of long decimals meaning small numbers, similar to decimal conceptions of school students (Stacey & Steinle, 1998).

Table 5
Adding one-quarter and three-fifths

Response	Inferred Conception
45	Visually logical from an approximation of one-quarter as one-fifth
39	Multiply numerators and add denominators
49	Add numerators and add denominators
320	Multiply numerators and multiply denominators
1720	Foundation content knowledge likely

Table 6
Ordering weights of trampers' packs

Response	Frequency	Inferred Conception
19.08; 19.098; 19.2; 19.2005	73%	Foundation knowledge likely
19.2005; 19.098; 19.08; 19.2	10%	Long decimal means small number
19.08; 19.098; 19.2005; 19.2	5%	Long decimal means small number applied to 19.2005 and 19.2
19.2005; 19.2; 19.098; 19.08	4%	Interpreting order as larger to smaller
19.2; 19.08; 19.098; 19.2005	1%	Short decimal means small number
Other	8%	Not interpreted

The question shown in Table 7 is an example of an adaptive assessment item that we identified as equivalent to the indicator question reported in Table 6. This parallel question was designed to find whether pre-service teachers could order decimal numbers correctly. Specifically, the parallel question was worded, "Four 100m runners recorded their fastest times in different ways. Here are their times (in seconds) 11.54, 11.087, 11.09, 11.6. Which of the following gives their times ordered from smallest to largest?" The parallel question was at Step 5 of the adult numeracy framework and probably had an item difficulty approximately the same as the indicator question, although a correct response in the adaptive assessment could have been selected using the conception "long decimal means small number." From these findings, we infer that at least 73% of the 2010 cohort was performing at Step 5 or above on the adult numeracy framework for the topic of ordering of decimals.

Table 7
Ordering sports times

Response	Inferred Conception
11.6, 11.09, 11.54, 11.087	Digits after the decimal point read as whole numbers (i.e., 6, 9, 54, 87) or all digits read as whole numbers (i.e. 116, 1109, 1154, 11087)
11.087, 11.54, 11.09, 11.6	Digits after the decimal point read as whole numbers and ordered largest to smallest (i.e., 87, 54, 9, 6)

Continued next page.

Response	Inferred Conception
11.09, 11.6, 11.54, 11.087	Digits after the decimal point read differentially (i.e., reading 09 smaller than 6, but 54 smaller than 87
11.54, 11.6, 11.087, 11.09	Digits after the decimal point read as numbers with a place value of ten (i.e., 54, 60, 87, 90)
11.087, 11.09, 11.54, 11.6	Foundation content knowledge likely

4. Matched questions about the volume of cuboids. Table 8 shows pre-service teachers' responses to an indicator question intended to reveal foundation knowledge of calculating the volume of cuboids. Specifically, the indicator question was worded, "Another cake tin is a cuboid (i.e. it has straight sides at right angles). The tin is 40cm long, 20cm wide and 10cm high. What is the volume of the tin in cubic centimetres?" Thirty-seven percent of pre-service teachers gave the correct numerical response, although of these, 8% omitted units or wrote incorrect units but were marked as correct.

Table 8
Volume of a cake tin

Response	Frequency	Inferred Conception
8,000cm3	29%	Foundation knowledge likely
800 (variety of units)	12%	Multiplying length and width – thinking multiplicatively in two dimensions; or procedural error in multiplying powers of ten
8000 (incorrect units or no units)	8%	Foundation knowledge possible
80 (variety of units) as length	5%	Volume must be same order of magnitude
70 (variety of units) in three dimensions	5%	Lengths of sides added - thinking additively
Other	22%	Not interpreted
No answer	19%	Uncertain how to respond

The question shown in Table 9 is an example of an adaptive assessment item that we have identified as equivalent to the indicator question reported in Table 8. This parallel question was designed to find whether pre-service teachers could calculate the volume of a rectangular prism. Specifically, the indicator question was worded, "A square sand pit has internal side lengths of 1.7m. You want to fill it with sand to a height of 0.2m. How many cubic metres of sand must you purchase?". The parallel question was at Step 6 of the adult numeracy framework and probably had an item difficulty approximately the same as the indicator question. From these findings we infer that approximately 37% of the 2010 cohort was performing at Step 6 on the adult numeracy framework for the topic of calculating the volume of cuboids.

Table 9
Volume of a sand pit

Response	Inferred Conception
3.6	Lengths of sides added - thinking additively in three dimensions
0.34	Multiplying length and depth - thinking multiplicatively in two dimensions
2.89	Multiplying length and width - thinking multiplicatively in two dimensions
0.578	Foundation content knowledge likely
4.913	Length cubed - thinking multiplicatively in three dimensions

Discussion

The majority of the pre-service teachers in this study did not demonstrate the foundation content knowledge needed to meet the professional standards of a practising teacher. In 2010, less than half of pre-service teachers passed the conventional test that was designed to assess general mathematics content knowledge and which included questions we considered to assess foundation content knowledge. Similarly, in 2011 less than half the pre-service teachers scored above the threshold mark in the adaptive assessment that was designed as an assessment of adult numeracy, which also included questions we considered to assess foundation content knowledge. These findings are in concordance with similar studies of pre-service teachers (Afamasaga-Fuata'i et al., 2007; Senk et al., 2012; Tobias & Itter 2007).

Despite the different forms and design intentions of the conventional and adaptive assessments, we were able to make comparative observations and offer inferences that were informed by the mathematics education literature (Hart, 1981; Stacey, et al., 2009; Ryan & MacCrae 2005/6; Senk, et al., 2012). We drew on this literature to identify matched questions for assessing the foundation content knowledge in both cohorts of participants in our study. Due to space considerations, we reported the results of four of the twelve questions in detail. We demonstrated that pre-service teachers' overall results on the conventional test corresponded fairly well to scores on the indicator questions. However, although only eight pre-service teachers with a high indicator question score failed the test, 28 pre-service teachers with a low indicator question score passed the test. This finding suggests that conventional tests designed with a broad range of questions at a variety of levels of difficulty may not be reliable gauges of pre-service teachers' foundation content knowledge. The adaptive assessment tool was designed to measure the level of numeracy of adult learners in New Zealand. We established that the adaptive assessment tool included many questions that were equivalent to the indicator questions that assessed foundation content knowledge in the conventional test. Our examination of these parallel indicator questions and the pre-service teachers' responses to them, suggested that achieving Step 6 on the TEC adult literacy and numeracy learning progression was comparable with correctly answering the indicator questions in the conventional test. Because of this comparability and because the adaptive assessment tool provides an estimate of the uncertainty of each pre-service teacher's score, the adaptive assessment tool may be a more reliable gauge of foundation content knowledge than the conventional test.

The adaptive assessment tool has a number of other advantages over conventional testing. Pre-service teachers do not need to be assessed at the same time and the assessment can be administered anywhere there is internet access. Therefore, large numbers of pre-service teachers can be assessed in flexible ways without disrupting timetables. Second, the adaptive assessment tool simplifies invigilation because each user is given a different selection of questions from a large item bank. Third, the use of Rasch analysis means that there can be consistency between pre- and post-assessments, across years, and among institutions because each pre-service teacher's raw score is converted to a location on a common measurement scale. We suggest that the considerable advantages of the adaptive assessment tool warrant further exploration of its use in initial teacher education, and possibly further development to tailor a version to the specific needs of pre-service teachers.

The use of Rasch analysis in the construction of assessment tools and analysis of assessment data has facilitated a number of innovations in the assessment of pre-service teachers (e.g., Ryan & McCrae, 2005/6; Ryan & Williams, 2007; Senk et al., 2012). Senk et al. identified clusters of assessment items of similar difficulty on their scale, which provided qualitative descriptions of the mathematics content knowledge that pre-service teachers were likely to demonstrate at those points. These descriptions assisted us in identifying indicator questions for assessing foundation content knowledge. In contrast to the assessment used by Senk et al., however, the adaptive assessment tool made use of Rasch analysis to establish the difficulty of an assessment item from the performance of a population of previous users. As a result, a pre-service teacher using the adaptive assessment tool was presented with questions appropriate to his or her ability, as estimated by his or her responses to earlier questions.

Ryan and her colleagues (Ryan & McCrae, 2005/6; Ryan & Williams, 2007) also used results of the Rasch analysis of a conventional test to identify questions where a pre-service teacher was performing at a level below his or her ability. Pre-service teachers were encouraged to interact with this diagnostic information in order to understand and develop their own knowledge. The use of diagnostic information is part of an emerging growth orientation to the design and conduct of research about pre-service teacher mathematics content knowledge (Ryan & Williams, 2011). In a manner similar to the tool of Ryan and her colleagues (Ryan & McCrae, 2005/6; Ryan & Williams, 2007), the adaptive assessment tool provides specific feedback to users on which questions are answered correctly and incorrectly. In contrast to the tool developed by Ryan and her colleagues, which establishes an ability score based on a fixed set of questions, the adaptive assessment tool tailors the questions it generates to the ability level of the pre-service teacher. Therefore, the results of the adaptive assessment may be more closely aligned to a pre-service teacher's immediate learning needs. A limitation of the adaptive assessment is that it was designed to assess knowledge of number and measurement only. However, we have argued that number and measurement are substantial components of the mathematics content knowledge needed for teaching in primary school settings.

We have narrowed the definition of mathematics content knowledge that pre-service teachers bring with them to the start of their initial teacher education programmes and termed it foundation content knowledge, to emphasise that it is the underlying knowledge that can be built on in pedagogical courses and practicum experiences. Increasingly, accrediting bodies are requiring initial teacher education providers to ensure that pre-service teachers have a sufficient base upon which to build the knowledge required to meet the professional standards of a practising teacher (Stephens, 2003). In New Zealand, however, it is not clear what this base knowledge should be and furthermore, it is not clear how such base knowledge should be assessed. We suggest that foundation content knowledge is a useful concept for describing the sufficiency and type of mathematics content knowledge that pre-service teachers must have as a base from which to build and from which to connect further knowledge of teaching.

Conclusion

We situated this study in relation to a growing body of literature that conceptualises the knowledge required for teaching mathematics as multi-dimensional and contextually situated (Bobis et al., 2012; Delaney et al., 2008; Fennema & Franke, 1992; Hill et al., 2008; Shulman, 1986, 1987; Rowland & Ruthven, 2011). Within this paradigm, mathematics content knowledge is regarded as crucial for the effective teaching of mathematics. While there has been research interest in developing and investigating theoretical models of the various forms of knowledge required for teaching, less attention has been paid to the need for research on the mathematics content knowledge that pre-service teachers bring with them into their initial education programmes. In this study we have sought to contribute the growing mathematics education research-base that is enhancing our understanding of mathematical knowledge for teaching.

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