Mathematics Teacher Educators Teaching to Teach: Tasks and Formative Purposes in Initial Teacher Education

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The study reported in this article examined mathematics teacher educators who teach didactics of mathematics in Uruguay. It analysed the tasks they selected and the purposes for which these were used in initial teacher education. The study involved twelve mathematics teacher educators from three teacher-education institutes and employed an in-depth qualitative case-study design. Data were collected through semi-structured interviews, classroom observations, and task exemplars used by the teacher educators. The findings indicate a wide range of tasks aligned with categories proposed in the literature—including reflecting on similarities and differences, designing and solving problems, learning through the study of practice, selecting resources, and identifying barriers to learning—yet with a low prevalence of tasks that address the beliefs of prospective mathematics teachers. The tasks served diverse formative purposes, including task design and redesign, reflection on students' thinking, lesson planning, managing interactions, resource analysis, types of proofs, and the languages present in the mathematics classroom. While most tasks aligned with established categories, some did not fit adequately—for example, tasks oriented towards the construction of the object to be taught. Additionally, broader purposes emerged that transcend immediate practice, viewing teacher education as a space for preparing prospective mathematics teachers to become critical practitioners.

Keywords · mathematics teacher educators · teaching didactics of mathematics · task themes

> In doing this kind of work, I often find that the theories developed independent of practice are either wrong or dangerously incomplete. I now invite you into the most intriguing of places, the valley where theory and practice intersect.
> Lee Shulman (2004, p. 506)

Introduction

Mathematics teacher education is a significant area within mathematics education research—also referred to in some traditions as didactics of mathematics (DM). In recent years, research on the profile and preparation of mathematics teacher educators (MTEs) has attracted particular attention (Beswick & Chapman, 2020; Castro & Li, 2014; Goos & Beswick, 2021). Along these lines, several recent studies have examined models of teaching and the impact of MTEs' practices on those in training (Escudero-Ávila et al., 2021; Marshman, 2021; Ochoviet & Olave, 2017; Ticknor, 2012). Studies such as Appova and Taylor (2019) and de los Ángeles et al. (2022) analysed the kinds of opportunities for reflection on classroom practice offered to prospective mathematics teachers (PMTs) during their preparation and identified the limited connections between mathematics teaching in higher education and the teaching that PMTs will undertake in their own classrooms. Jaworski and Huang (2014) noted that research on the knowledge required of those who teach DM has received less attention than research on MTEs who teach mathematics, and that it constitutes an emerging line of inquiry in mathematics education. In this study, the focus is specifically on MTEs who teach DM, understood—following Escudero-Ávila et al. (2021)—as those who engage simultaneously in two processes: (a) helping PMTs to understand mathematics as an object of teaching and learning, and (b) supporting them in transforming mathematical ideas and procedures into forms that are accessible to primary and secondary school students.

In these two processes, as well as in professional development programmes, tasks are a key resource for promoting learning about the work of teaching mathematics (Watson & Mason, 2007; Zaslavsky & Sullivan, 2011). Designing challenging tasks by the MTE requires flexibly mobilising the knowledge needed to support the mathematical and pedagogical development of PMTs (Chapman, 2021; Chick & Beswick, 2018). Beliefs, conceptions, and experience also shape the decisions the MTE makes (Chick & Beswick, 2018; Escudero-Ávila et al., 2021; Peled & Suzan, 2011). In this regard, Zazkis and Mamolo (2018) argued that the qualities of MTEs' knowledge that enable the design of tasks aimed at supporting teacher development—and how such qualities develop and are manifested in practice—remain an open area of inquiry.

Recent literature offers valuable examples of different families of tasks used in teacher education and professional development, with a predominant emphasis on their design and redesign rather than on criteria for selection and sequencing (Liljedahl et al., 2007; Markle & McGarvey, 2025; Peled & Suzan, 2011; Runesson et al., 2011; Silver et al., 2007; Swan, 2011; Watson & Bills, 2011; Zazkis & Mamolo, 2018). These studies also reported diverse effects of experimentation with tasks, such as the development of reflection and noticing (Vondrová, 2020), the development of sensitivity to students' thinking and the identification of critical aspects of learning particular content (Runesson et al., 2011; Watson & Bills, 2011), shifts in teachers' beliefs and conceptions (Swan, 2011), and the reorganisation of the mathematics mobilised in practice (Silver et al., 2007).

This study therefore seeks to contribute to two areas of mathematics education that remain under-explored: what MTEs teach (Goos & Beswick, 2021), and what tasks they select (Markle & McGarvey, 2025). In addition, unlike studies that trial tasks in experimental or design contexts (e.g., Silver et al., 2007; Swan, 2007), this study does not seek to trial new tasks but rather to observe what occurs in the ordinary classrooms of those who teach DM in teacher education programmes.

This article reports the findings of a study conducted in initial mathematics teacher education in Uruguay, specifically within Didactics of Mathematics/Teaching Practice courses in the second, third and fourth years of the *Profesorado de Matemática* degree programme. These courses are led by MTEs, who address theoretical aspects of DM and issues related to teaching practice. In addition, the MTEs supervise the annual teaching practice undertaken by PMTs in secondary schools. The study addressed the following questions:

(1) *What task themes do MTEs address through the tasks they propose in their courses, with the aim of preparing prospective mathematics teachers to teach mathematics?*

(2) *In what ways, from the perspective of the MTEs, are the tasks connected to the overarching purpose of teaching how to teach mathematics?*

Conceptual Framework

According to Escudero-Ávila et al. (2021), teaching how to teach mathematics requires the MTE to design and implement tasks "which fulfil the specific needs of the elements of the teaching profession he or she aims to develop. These tasks, and how they are used, have an impact on prospective teachers' learning" (p. 33). This fosters the development of the knowledge mobilised in teaching, professional identity, and professional skills (da Ponte, 2012). The authors add that an MTE who teaches DM should not only be familiar with a wide repertoire of activities that enable the development of the three elements identified by da Ponte but also understand the potential and limitations of the tasks proposed, together with their implications for learning and the future professional work of those in training. Among the various activities an MTE should be able to implement, Escudero-Ávila et al. (2021) mentioned the facilitation of discussion groups, video analysis, the use of narratives, and case studies, among others. Moreover, such activities should be developed in an appropriate environment that enables the construction of the professional knowledge required for teaching mathematics. In this regard, these authors noted that the MTE who prepares mathematics teachers should possess knowledge of the pedagogical content knowledge that prospective teachers need to develop. They further stated that the MTE who prepares mathematics teachers is "the person responsible for managing and promoting the construction of teachers' professional knowledge" (p. 28) and, therefore, should have a deep understanding of the Mathematical Knowledge for Teaching (Ball et al., 2008), its components, and the contributions of research on this topic.

For example, in the work of the MTE who teaches DM, their mathematical knowledge is brought to bear, though not in the same way as in their role as a secondary-school teacher (Beswick & Goos, 2018), because this knowledge now becomes the object of teaching; that is, it is not a matter of mobilising specialised content knowledge (Ball et al., 2008) but of teaching it in the teacher education classroom, which requires a particular transformation of that knowledge so that PMTs can learn it. Escudero-Ávila et al. (2021) noted that:

> Indeed, we have found that one of the basic characteristics of an MTE's knowledge lies in the sheer density of the connections which they are able to make between the diverse elements of knowledge relating to mathematics as the object of teaching and learning. And it is these connections that allow the MTE to create a suitable environment for prospective teachers to construct their own rich and integrated knowledge. (p. 24)

Escudero-Ávila et al. (2021) added that, in addition to knowledge, MTEs hold conceptions and beliefs that shape their personal view of what PMTs should learn and, in turn, determine the preparation they provide. Additionally, they observed that MTEs

> … have their own ideas about how best to help their students on their journey to becoming teachers, that is, ideas about how people learn (what he or she understands should be learnt) and how best to collaborate in the process (which determines their personal model of mathematics teacher education). (p. 25)

Rojas et al. (2021) conceptualised MTEs who prepare mathematics teachers as "agents tasked with helping pre-service teachers improve their skills and facilitating the teaching of mathematics" (p. 368). Rojas and Deulofeu (2015) argued that one of the essential tasks of MTEs who prepare mathematics teachers is to promote activities in the university classroom so that PMTs learn how to teach mathematics, placing particular emphasis on the relationships between theory and practice—an emphasis that is central to the work of an MTE teaching DM. Whereas other frameworks, such as that of Chick and Beswick (2018), tend to begin by foregrounding the pedagogical content knowledge to be promoted and then propose task examples aligned with each knowledge category, the present study adopted a different point of departure. In this instance, the task was taken as the starting point from which to identify what the MTE sought to teach or promote through its design and enactment.

This paper therefore focuses on the design of tasks intended to prepare PMTs to teach mathematics. In this vein, Zaslavsky (2008) noted a consensus among MTEs that learning occurs when learners engage in tackling tasks. Several authors argued that the quality of learning depends on the tasks that are set and on how they are implemented in the classroom (Hiebert & Wearne, 1997; Watson & Mason, 2007; among others). Building on these ideas, Zaslavsky (2008) argued that, in teacher education, tasks play an essential role in teachers' learning insofar as they promote knowledge about the teaching of mathematics:

> Ideally such tasks provide a bridge between theory and practice, and serve to challenge, surprise, disturb, confront, extend, or provoke examination of alternatives, drawn from the context of teaching. In this context, tasks are seen as problems or activities that, having been developed, evaluated and refined over time, are posed to teacher education participants. (p. 94)

From this perspective, Zaslavsky (2008) underscored the centrality of the MTE's role as a designer of tasks and learning environments that foster learning about the work of teaching mathematics. Zaslavsky further highlighted that, while designing and implementing such tasks, the MTE learns from their practice, which contributes to their professional development.

To understand the role of the MTE in supporting the learning of PMTs, as well as the knowledge the MTE draws on, Zaslavsky and Sullivan (2011) proposed eight unifying task themes that, in their view, encompass the competences and knowledge demanded by mathematics teacher education (Table 1). These themes provide dimensions for analysing the types of tasks proposed by MTEs who teach DM to PMTs in this study.

Table 1
Task Themes for the Preparation of Secondary-school Mathematics Teachers (Zaslavsky & Sullivan, 2011)

Theme	Description
(1) Developing adaptability	Adapting and varying the wording of questions and to modify tasks, with the aim of exploring alternatives that improve teaching proposals, as well as the ability to adapt existing resources to the formative aims. It also entails being prepared to respond to contingencies. The tasks proposed should demand adaptability to pedagogical, curricular, and mathematical requirements.
(2) Fostering awareness of similarities and differences	Developing the ability to notice similarities and differences across problems, contexts, errors, classroom situations, and mathematical objects, to classify them, detect patterns, establish curricular connections, and define pedagogical approaches. The tasks proposed should likewise require engagement with pedagogical, curricular, and mathematical perspectives.
(3) Coping with conflicts and dilemmas	Preparing teachers to make decisions in complex and conflictual situations and to manage uncertainty: to anticipate conflicts, design tasks that involve conflict for students, and identify inconsistencies in curriculum documents. The tasks proposed should likewise require engagement with pedagogical, curricular, and mathematical perspectives.
(4) Designing and solving problems for use in mathematics classroom	Designing problems and working with them in class; incorporating problems that admit multiple solutions or solution strategies; increasing the open-endedness of textbook problems; and using games as a context for problem solving.
(5) Learning from the study of practice	Accessing real or simulated practice and promote critical, reflective analysis of it—for example, the analysis of classroom video, exemplary lessons, lesson study, collaborative lesson planning, observation, and collective discussion and reflection on a taught lesson—as well as case studies that enable reflection on pedagogical, curricular, or mathematical challenges in practice.
(6) Selecting and using (appropriate) tools and resources for teaching	Selecting and using tools such as textbooks, supplementary readings, manipulatives, measuring and construction instruments, graphing calculators, and other technological environments. Tools are understood broadly to include different kinds of resources, including human and cultural resources. The tasks promote the selection, design, implementation, and analysis of the potential of such tools, among other aspects.
(7) Identifying and overcoming barriers to students' learning	Addressing barriers such as epistemological aspects of mathematics, cultural factors, gender, familial expectations, language, physical and other disabilities, and socio-economic factors. Identifying these barriers may be approached from a pedagogical perspective (e.g., gender, language, socio-economic issues) or a mathematical perspective (e.g., task types, representations, excessive formalism). The tasks aim to develop sensitivity to students' thinking and ideas.
(8) Sharing and revealing self, peer, and student dispositions	Recognising that teaching and learning have an attitudinal dimension that shapes both. It includes beliefs about the nature and utility of mathematics, beliefs about learning mathematics and one's own capacity to learn it; self-regulatory processes related to learning; and attitudes towards mathematics, such as enjoyment and mathematics anxiety.

This list of themes is not exhaustive; that is, it has the potential to be extended through further research—an aspect this study seeks to explore. Moreover, the categories are not mutually exclusive. For example, when designing a problem (Theme 4), an MTE might start from an existing textbook task and adapt it (Theme 1) to turn it into an open-ended problem (Zaslavsky, 1995), so that it is better suited to the diversity of students in the classroom (Sullivan & Clarke, 1992). Such overlap, among other possible forms, offers a rich tapestry that can make visible the complexity of MTEs' knowledge (Escudero-Ávila et al., 2021).

Zaslavsky (2008) argued that designing productive tasks for use in the teacher-education classroom is challenging for MTEs because, unlike mathematics teachers, those who teach DM have access to very few resources—such as textbooks or teachers' guides—to support their planning. Although they can draw on research-based materials, theoretical handbooks, and their teaching experience, task design nevertheless entails a distinctive process of development.

Method

Participants and Data Collection

This study adopted a qualitative, in-depth case-study design (Patton, 2002). The participant group comprised 12 MTEs (MTE1, MTE2, MTE3, etc.) with 3 to 21 years of teaching experience from three teacher-education institutes in Uruguay. All participants held a secondary mathematics teaching degree and postgraduate qualifications in mathematics education or education; none held a degree in mathematics. They taught DM within an initial mathematics teacher education programme preparing prospective teachers to teach mathematics in secondary schools.

Data were collected across the participating MTEs' DM/Teaching Practice courses through semi-structured interviews, classroom observations conducted during their regular teaching, and analysis of task exemplars used in those classes to support PMTs in learning how to teach mathematics. The lessons observed were audio-recorded and subsequently transcribed. Two semi-structured interviews were conducted with each MTE—one prior to the lesson observed and another afterwards—which were video-recorded and then transcribed. In this article, quotations from the interviews are attributed to the responding participant using the participant code (e.g., MTE5) and the interview code (I1 or I2). For example, [MTE5, I2].

The study was carried out in the usual contexts of teacher education, engaging with the practices and materials that MTEs routinely use in that setting. No specific sequence of tasks was designed or trialled; instead, classes were observed as they ordinarily unfold in teacher education.

The information obtained enabled triangulation—not with the aim of verifying the truth of participants' statements, but because inconsistencies provide opportunities to understand the complexity of the issue under study (Mok & Clarke, 2015). The phases were organised as follows:

1. Preliminary semi-structured interview (I1) with participants. This study explored the overarching aims of the Didactics of Mathematics/Teaching Practice courses and how MTEs conceive of teaching others how to teach; strengths and weaknesses in this endeavour; and examples of activities they bring to the classroom together with their intended purposes.

2. Classroom observation. Focus on task implementation, the MTE's foci of attention, the connections established, the knowledge for the teaching of mathematics that is promoted, the types of classroom interaction, etc.

3. Post-observation semi-structured interview (I2) to elicit the MTE's perspective on the implemented lesson, including a tentative title for the task employed and its connection to the aim of teaching PMTs to teach mathematics.

Dimensions to Guide Lesson Observation

MTEs were invited to be observed teaching a class in which they implemented a task of their own choosing, with the explicit aim of "teaching how to teach mathematics", as each MTE understood this aim in terms of its dimensions and personal approach. The task did not need to be original to that lesson; they could draw on a task implemented in previous years or with other groups. The duration of the observation was determined by each MTE, with a minimum of 45 minutes. The observation focused on the MTE's practice rather than on the PMTs' thinking or contributions, while attending to how the MTE managed these. The classroom observation considered the following dimensions (Table 2).

Table 2
Dimensions to Guide Lesson Observation

Dimension	Description
Treatment of DM constructs and explicit reference to their authors	The MTE explains and elaborates constructs from the field of DM, citing the associated authors and making explicit how these constructs relate to the topic under discussion.
Connections with classroom practice	The MTE elaborates with examples or cases drawn from practice, or with classroom resources and their adaptability to different contexts.
Connections between research and classroom practice.	The MTE supplements the topic under discussion with research that has addressed this topic or related topics.
Connections with secondary-school students' thinking	The MTE notes common errors, difficulties, or obstacles encountered by secondary-school students, illustrates these with examples, and/or discusses strategies for addressing them.
Connections with the curriculum	The MTE maps the topic under consideration onto the secondary-school curriculum (and the primary curriculum, where appropriate) and establishes connections between this topic and others.
Eliciting and managing PMTs' contributions	The MTE elicits contributions from PMTs and incorporates them into the development of the lesson, fostering the negotiation of meaning. The MTE attends to PMTs' questions and uses them as prompts for collective discussion.
Beliefs and attitudes towards mathematics and its teaching and learning	The MTE enables beliefs and attitudes towards mathematics and its teaching and learning—both those of PMTs and of secondary-school students—to be made explicit during the lesson, and fosters reflection on these and on their impact on the teaching and learning of mathematics.
The MTE enriches PMTs' contributions by providing additional information or classroom-based examples	The MTE extends or complements PMTs' contributions by situating them within one of the dimensions of the Mathematical Knowledge for Teaching.
The MTE provides pedagogical or mathematical clarifications regarding PMTs' contributions	Where necessary, the MTE clarifies or qualifies the concepts, ideas, or beliefs expressed by PMTs, with the aim of elucidating mathematical or pedagogical issues and addressing misconceptions.

Criteria for Categorising Task Themes

To categorise the theme(s) addressed by the task implemented in the lesson observed, following Zaslavsky and Sullivan (2011), the initial analysis involved a broad categorisation that considered all the themes the task touched upon, even where the connection was weak. To refine this initial categorisation, and to provide a perspective complementary to the thematic connections interpreted by the researcher, each MTE was asked to propose a thematic title for the task used in class, indicating the focus they considered most salient. Subsequently, this thematic focus was cross-checked against the Zaslavsky and Sullivan (2011) categorisation to determine whether it was already encompassed by their themes or warranted extending the categories to include a new theme, and each task was then assigned to a single theme in order to obtain a mapping of the principal thematic foci.

Findings and Discussion

Task Themes in the Lessons Observed

Twelve lessons within the Didactics of Mathematics/Teaching Practice courses were observed, in which MTEs had been asked in advance to implement a task of their own choosing with the explicit aim of "teaching how to teach mathematics" (as each MTE understood that aim).

The lessons observed did not differ substantially in how classroom interactions were organised: generally, there was dialogue among PMTs and between PMTs and the MTE around a focal question for discussion. No lessons followed a "tell-and-show" approach; rather, MTEs worked with non-routine situations that fostered reflection on teaching. In this respect, although participants varied in their experience as MTEs, they displayed characteristics of expert MTEs, as described by Appova and Taylor (2019). However, there were differences in the task themes, in their specific formative aims, and in the scope of the discussions.

The thematic categories identified are, in general, consistent with those proposed by Zaslavsky and Sullivan (2011) for mathematics teacher education. Most tasks addressed more than one theme (Table 3). This aligns with the complexity of MTEs' teaching, since supporting PMTs' learning often requires attending to multiple formative requirements, such as those "necessary for prospective teachers to develop their professional identity and skills and to construct their professional knowledge" (Escudero-Ávila et al., 2021, p. 36). The data indicate that each MTE planned tasks around a dominant theme alongside additional, closely related emphases. Viewed across a course, this layering may help sustain and deepen attention to a diversity of themes as PMTs encounter them in different tasks and contexts.

Table 3
Task Themes by MTE

Task themes in the lessons observed	MTE											
	1	2	3	4	5	6	7	8	9	10	11	12
(1) Developing adaptability	X	X	X		X	X	X	X		X		
(2) Fostering awareness of similarities and differences			X	X	X	X		X	X	X	X	X
(3) Coping with conflicts and dilemmas		X	X					X	X	X		
(4) Designing and solving problems for use in mathematics classroom	X	X	X	X	X	X	X	X		X	X	
(5) Learning from the study of practice		X	X				X	X		X		
(6) Selecting and using (appropriate) tools and resources for teaching		X	X	X	X		X	X	X	X	X	
(7) Identifying and overcoming barriers to students' learning		X	X			X	X	X	X	X		
(8) Sharing and revealing self, peer, and student dispositions							X	X	X	X		

The most frequently addressed themes in the lessons observed were *fostering awareness of similarities and differences; designing and solving problems for use in mathematics classroom; and selecting and using (appropriate) tools;* and *resources for teaching*. Substantial attention was also devoted to *developing adaptability* and to *identifying and overcoming barriers to students' learning.*

These were followed by *coping with conflicts and dilemmas; learning from the study of practice*; and *sharing and revealing self, peer, and student dispositions*.

With the aim of reporting the identified themes with greater specificity, in the post-observation interview MTEs were asked to propose a tentative title for the task they had implemented, bearing in mind the primary objective the MTE sought to address with PMTs. The titles they provided suggested that all the MTEs designed a task with a main purpose in mind, without precluding in-class engagement with related themes. These contributions were considered alongside the researcher's observations to state, as clearly as possible, what was addressed in the lesson. Accordingly, given the specificity of the task proposed, the distribution of themes is shown in Table 4.

Table 4
Categorisation of Task Themes for Tasks Implemented in the Lessons Observed, According to Their Main Focus

Task themes in the lessons observed	MTE and task title proposed in the post-observation interview
(1) Developing adaptability	–
(2) Fostering awareness of similarities and differences	MTE9: Reflecting on mathematical proof and analysing different types of proof
	MTE12: Analysing language and linguistic practices in the mathematics classroom
(3) Coping with conflicts and dilemmas	–
(4) Designing and solving problems for use in mathematics classroom	MTE1: Designing or redesigning tasks
	MTE6: Analysing the cognitive demand of tasks, anticipating, and guiding classroom interactions
	MTE11: Increasing the cognitive demand of tasks set for students
(5) Learning from the study of practice	MTE2: Lesson simulation as a strategy for improving prospective mathematics teachers' practice
	MTE3: Constructing the object to be taught and its dimensions
	MTE8: Orchestrating classroom activities that place students centre stage and support the construction (rather than transmission) of knowledge
	MTE10: Imagining practice, in particular orchestrating whole-class discussion to foster interaction in the mathematics classroom
(6) Selecting and using (appropriate) tools and resources for teaching	MTE4: Reflecting on and analysing the pedagogical potential of a digital resource
	MTE5: A guide to sourcing resources and integrating them across subjects
(7) Identifying and overcoming barriers to students' learning	MTE7: Developing an interest in understanding students' thinking and the ability to anticipate possible responses
(8) Sharing and revealing self, peer, and student dispositions	–

As explained earlier, Table 4 places each task in a single category according to its primary focus; on that basis, Themes 4 and 5 were the most prevalent. It is conjectured that this may be because the DM courses taught by the MTEs include a practical component: alongside the theoretical component, all PMTs were simultaneously undertaking a teaching practicum in a secondary school. This may help to explain the MTEs' concern to connect their teaching with classroom practice, consistent with Rojas and Deulofeu (2015), and, at the same time, to support the development of a fundamental professional skill for all teachers: designing or selecting problems to set for their students in the mathematics classroom.

There are no tasks whose specific objective is *developing adaptability, coping with conflicts and dilemmas*, or *sharing and revealing self, peer, and student dispositions*.

Developing adaptability is fundamental for adjusting teaching when expected outcomes are not achieved and, as Zaslavsky and Sullivan (2011) note, because it encompasses the development of skills for handling contingencies in lessons: "there is a need for teachers to be prepared to make active responses that cannot be planned in advance" (p. 3). With respect to *coping with conflicts and dilemmas*, greater emphasis on this theme is warranted; in particular, Leikin's (2021) recommendation regarding the importance of designing tasks that combine mathematical challenge with the psychological and didactic challenges that teachers face in their daily work should be heeded. As for *sharing and revealing self, peer, and student dispositions*, this theme merits particular attention from MTEs, given that teachers' beliefs permeate their teaching (Beswick, 2005; Liljedahl et al., 2021; Philipp, 2007) and should be explicitly addressed during initial teacher education.

Illustrative Tasks by Main Theme

Each theme evidenced in the data is illustrated with a task drawn from the corresponding category.

Theme 2: Fostering awareness of similarities and differences

Theme 2 is illustrated by the task proposed by MTE9, which the participant titled "Reflecting on mathematical proof and analysing different types of proof." In the lesson observed, MTE9 led a discussion about the types of proof brought into secondary classrooms, juxtaposing "proofs that prove" with "proofs that explain" (Hanna, 1989). Figure 1 presents a proof by mathematical induction for the sum of the first *n* odd numbers, produced by an 11th-grade student. Questions in Part 1 are related to assessment and feedback practices; whereas questions in Part 2 are related to mathematics content knowledge. Part 3 (Figure 2) presents a proof without words by a student in the same grade. The proofs in Figure 1 and Figure 2 are accompanied by questions intended to prompt reflection. In Part 4 (Figure 2), PMTs are asked which of the two proofs they would choose to use in class.

Sebastián, a fifth-year student in the science track at a secondary school in Montevideo, submitted the following work to his teacher to prove that $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for every natural number n with $n \geq 1$.

I verify the initial case: $n = 1$ $\quad 1 = 1$

H) $n = h$ $\quad 1 + 3 + 5 + \ldots\ldots + 2h - 1 = h^2$
T) $n = h + 1$ $\quad 1 + 3 + 5 + \ldots\ldots + 2(h + 1) - 1 = (h+1)^2$

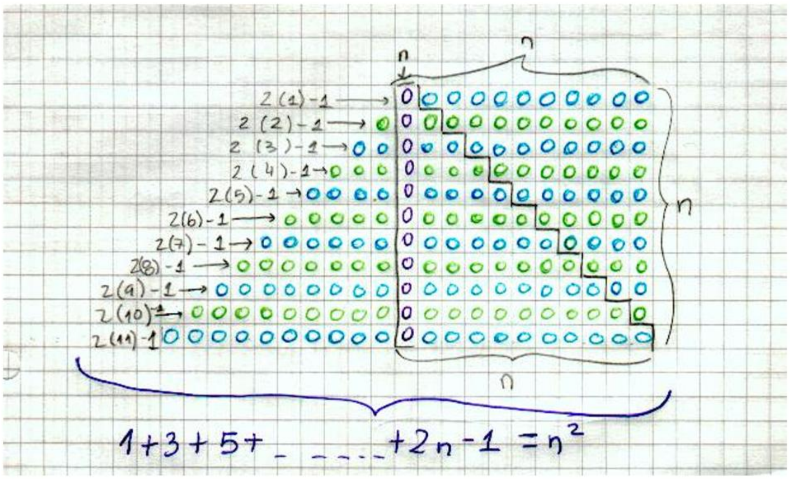
$1 + 3 + 5 + \ldots\ldots + 2h + 2 - 1 = (h + 1)^2$
$1 + 3 + 5 + \ldots\ldots + 2h + 1 = (h + 1)^2$
$\underbrace{1 + 3 + 5 + \ldots\ldots + 2h - 1}\, + 2h + 1 = (h + 1)^2$

$h^2 + 2h + 1 = (h + 1)^2$
$h^2 + 2h + 1 = h^2 + 2h + 1$ $\quad$ It holds

1) Mark and comment on Sebastián's work as if it had been submitted by one of your students. What mark would you award? What suggestions would you give the student?
2) Do you consider that Sebastián's work constitutes a proof? Why?

Figure 1. Task prompt (Parts 1 & 2), proposed by MTE9 (based on Ochoviet & Olave, 2011, p. 9).

3) Valentina, Sebastián's classmate, submitted the following work:



a) Mark and comment on Valentina's work as if it had been submitted by one of your students. What mark would you award? What suggestions would you give the student?

b) Do you consider that Valentina's work constitutes a proof? Why?

4) If you had to prove in class that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for every natural number n with $n \geq 1$, how would you proceed? Which of the two approaches presented above (Valentina's or Sebastián's) do you consider more appropriate? Why?

Figure 2. Task prompt (Parts 3 & 4), proposed by MTE9 (based on Ochoviet & Olave, 2011, p. 9).

In the post-observation interview, MTE9 explained why she implemented this task in class with the aim of teaching how to teach:

> Various studies, and my own experience—both as a teacher and in working with colleagues—suggest that teaching mathematics is often construed as teaching a set of rules, theorems, axioms and definitions. Knowing all of that is taken to be what one must do: to transmit what is already assembled, a ready-made mathematics. [MTE9, I2]

The participant added that, unless students are positioned in their future role as teachers and traditional classroom practices are questioned, they "will reproduce that way of teaching." [MTE9, I2] She emphasised that the task she proposed would enable PMTs "to decide which types of proofs they might choose to present to their students when they are teachers and to distinguish the value of each." MTE9's remarks foregrounded similarities and differences—here, between mathematical proofs. The task directs attention to students' thinking to prompt reflection on pedagogical approaches: Which types of proofs will be used in mathematics lessons?

Theme 4: Designing and solving problems for use in mathematics classroom

This theme is illustrated using the task proposed by MTE11, which was titled "Increasing the cognitive demand of tasks set for students." The MTE set a geometric task (Figures 3 & 4) that brought properties of quadrilaterals into play without the teacher having taught them beforehand. She sought to break with a common approach in geometry lessons— "tell-and-show" followed by application. The activity concluded with a student reading aloud from Smith and Stein (1998), which explains what low- and high-cognitive-demand tasks are. Drawing on the characteristics of low-cognitive-demand tasks, the MTE noted that one might instead have set a task that simply stated, "State the definition of a rhombus," to contrast its cognitive demand with that of the tasks used in the lesson.

I have drawn a quadrilateral on a sheet of paper hidden from view. Of this quadrilateral we know that it has: at least one pair of perpendicular sides, at least two equal angles, sides congruent in pairs, and a diagonal measuring 15 cm.
The aim is to construct a quadrilateral congruent to the hidden one.

Figure 3. Task prompt (Part 1), proposed by MTE11 (Individual work).

We are going to play the "Hidden Quadrilateral" game. Work with your classmates to design a list of four questions to identify the hidden quadrilateral.
As before: I will only answer yes or no, and you may not mention any polygon in the wording of your questions.

Figure 4. Task prompt (Part 2), proposed by MTE11 (Teamwork).

When asked in the post-observation interview about the objectives of this task, MTE11 indicated that understanding the cognitive demand required of students is part of the teacher's work. She also added that PMTs often take the tasks they use in classrooms from students' textbooks and that, drawing on what is studied in her course, they learn to modify them to raise the tasks' cognitive demand: "Although we can often create tasks from scratch, we also frequently adapt them, and it is good for them to be able to adapt them from that perspective." [MTE11, I2]. Furthermore, MTE11 emphasised the importance of PMTs experiencing task demands first-hand; for that reason, she set such tasks in her lesson for them to solve, so that PMTs realise it is not necessary to teach the definitions of the different types of quadrilaterals beforehand; rather, by solving the task, the definitions are brought into play.

Theme 5: Learning from the study of practice

Theme 5 is illustrated using the task proposed by MTE2, which consisted of a lesson simulation. The MTE titled it "Lesson simulation as a strategy for improving prospective mathematics teachers' practice." Prior to the simulation, MTE2 asked PMTs, working in teams, to solve a problem taken from a ninth-grade secondary-school textbook (Figure 5) and to plan how to address it for an eighth-grade class as a pre-class task, bringing their completed plans to the session. The rationale for using this problem in eighth grade rather than ninth was that strategies such as the use of systems of linear equations are not yet available in eighth grade; accordingly, PMTs had to devise more resourceful solution strategies aligned with the tools available to eighth-grade students. In this respect, the task also connects with Theme 1, as it requires adaptation.

Suppose two people are walking in the same direction towards point D. Mar starts from point A, which lies 0.5 m from O, and walks at a speed of 1 m/s. Leo starts from point B, which lies 2 m from O, and walks at a speed of 0.5 m/s. Does Mar catch up with Leo? If so, when?

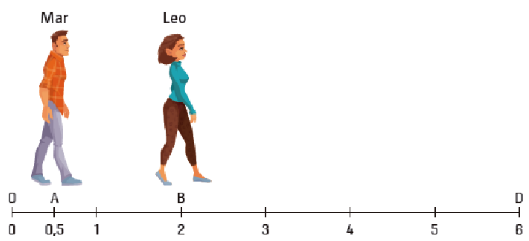


Figure 5. Task prompt proposed by MTE2 (Ochoviet & Olave, 2021, p. 50).

The methodology adopted was a lesson simulation which, based on what was observed, functioned as a rehearsal of practice (Kazemi et al., 2016): one PMT came to the front, acted as the teacher, and managed the lesson with peers enacting the role of students alongside the MTE. At points, the MTE paused the lesson to offer observations, pose questions, or make suggestions to foster critical reflection. The procedure was then repeated with another PMT, who played the role of teacher using their own lesson plan. The PMTs were able to appreciate different resources according to each PMT's planning: work at the board; concrete materials (e.g., a rope) to model the walk of the two characters in the problem; the use of technology (introduced by the MTE with GeoGebra); and different representational registers to model the phenomenon.

When asked how the task implemented in the lesson observed connected to the aim of teaching how to teach mathematics, MTE2 replied: "The task itself was intended to make students aware of the different teaching methodologies that a teacher may develop in the classroom." [MTE2, I2] MTE2 then added that "the idea of the simulation and the possibility of encountering difficulties or generating unforeseen circumstances helps them develop strategies that could facilitate a better-running lesson." In this latter point, the MTE refers to contingencies (Rowland et al., 2005) that may arise in the simulation of practice, thereby linking Theme 5 with Theme 1.

Theme 6: Selecting and using (appropriate) tools and resources for teaching

This theme is illustrated using the task employed by MTE6, which was titled "Reflecting on and analysing the pedagogical potential of a digital resource." He proposed four GeoGebra applets and asked PMTs to select one of them to analyse for which secondary-school year they would implement it and with what pedagogical purpose. The task prompt of the applet titled *Whimsical colors* reads: Which rule makes each of the colored "lights" turn on? Drag the vertices of the quadrilateral to find out. (Available at: https://www.geogebra.org/m/mbebv6h2) (Figure 6).

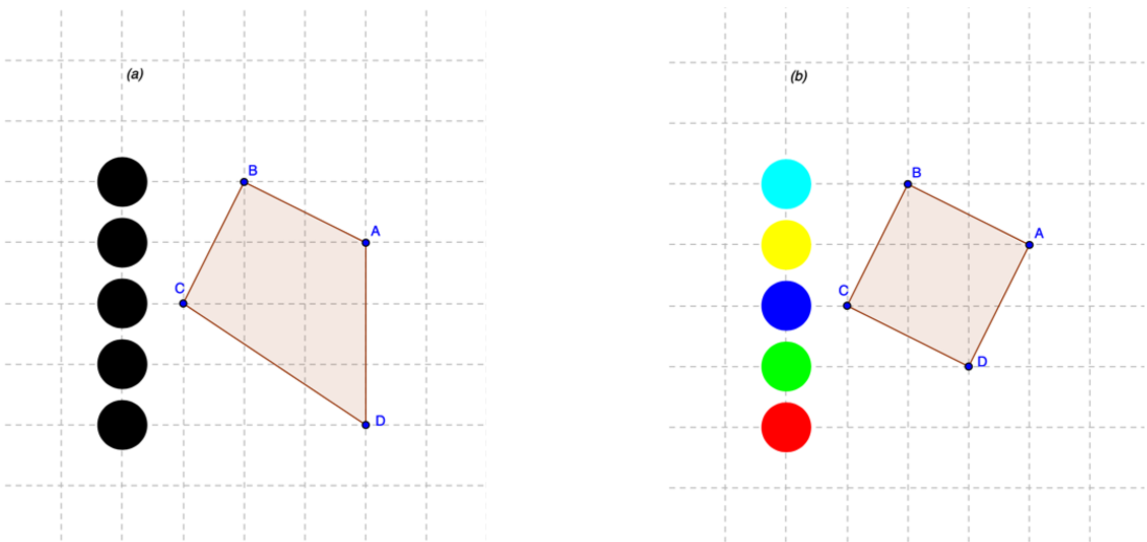


Figure 6. An applet offered by MTE6: *Whimsical colors*. (a) In this quadrilateral, all the lights (shown as black dots on the left) are off; (b) the quadrilateral is a square and all the lights are on.

When asked how this task connected to the aim of teaching how to teach mathematics, MTE6 replied that the intention was:

> To propose situations that elicit the kind of didactic reflection proper to the teacher: analysing a resource, identifying the mathematics that underpins it, the relationships, the representations brought into play, the possible interactions with students, etc.; all of that reflection which, in my view, a mathematics teacher should undertake is, based on personal opinion, perception and experience, something that prospective mathematics teachers find difficult. [MTE6, I2]

MTE6 added that this type of task helps PMTs to reflect as teachers and is useful to "exemplify how the planning process—the decision-making prior to the lesson—ought to be enriched when one plans a lesson." [MTE6, I2] The MTE makes explicit that personal opinions and experiences permeate teaching decisions (Escudero-Ávila et al., 2021); in the rationale offered, the choice of a specific resource is shown to be inseparable from an analysis of all the mathematical and pedagogical dimensions of the lesson.

Theme 7: Identifying and overcoming barriers to students' learning

This theme is illustrated by the task used by MTE7, which was titled, "Developing an interest in understanding students' thinking and the ability to anticipate possible responses." MTE7 provided activities based on Parodi (2016) that presented eighth-grade students' work on tasks involving different meanings of the equals sign (Figures 7 & 8). The MTE asked the PMTs to interpret the students' thinking and to analyse what might have led them to solve the tasks as they did. MTE7 also asked the PMTs to design tasks that could help these students overcome the difficulties evident in their work.

Consider the following task, taken from Parodi (2016), for use at the start of Year 9 in lower secondary education.

Write an equation that has 5 as a solution. Explain your reasoning.

1. Provide an a priori analysis. Indicate the mathematical ideas that are brought into play.
2. Specify the interventions you would make before and during the whole-class discussion.
3. Would you propose this task? With what aims?

Figure 7. Task prompt (Part 1), proposed by MTE7.

The following are Mateo's responses, as reported in Parodi (2016).

Write an equation that has 5 as a solution. Explain your reasoning.

$3\times5-x=5$

$3\times5-10=5$ (primero busco la ecuacion que de 5)

$3\times5-x=5$ (ahora le quito un número y lo represento como x)

What might Mateo have been thinking when giving that response?

Propose an activity to help improve Mateo's mathematical understanding of this topic.

Figure 8. Task prompt (Part 2), proposed by MTE7.

MTE7 justified the choice of this task as follows:

> I consider that, to teach mathematics, it is important not only to have a clear idea—of course—of what mathematical content is to be taught and to know that content in depth, but also to take an interest in what students think about that topic. [MTE7, I2]

The MTE's actions were interpreted as signalling an intention for PMTs to develop sensitivity to students' thinking, as well as skills for identifying critical aspects of the content to be taught (Runesson et al., 2011). The task aligns with these aims in that it requires anticipating responses, interpreting students' thinking, identifying specific difficulties—here, related to the meanings of the equals sign—and designing problems that could help students overcome them. The attitudinal component highlighted by the MTE accords with Jaworski (1994), who observes that sensitivity to students is one of the three central elements of teaching, alongside the notion of mathematical challenge and the management of learning. In this MTE's case, there appears to be a clear intention to cultivate such sensitivity, through the case of the student Mateo included among the proposed activities.

Emergent theme: Constructing the object to be taught

For the most part, the tasks proposed in the lessons observed could be placed within one of the categories proposed by Zaslavsky and Sullivan (2011). There was, however, one task that did not fit neatly within a single category of the framework's descriptors when classified according to its primary focus. This was the task employed by MTE3, which was titled "Constructing the object to be taught and its dimensions." Here, "object" refers to the mathematical concept/content to be taught, considered in terms of its key dimensions (e.g., representations, properties, typical tasks) and its connections with related concepts. This example reinforces the point made earlier that the set of theme categories is not exhaustive.

In the lesson observed, the MTE handed out sheets of paper to the PMTs and asked them to write down everything that came to mind when they heard the term "straight line." The MTE then distributed the following task (Figure 9).

> Solve the following activity and specify the purpose for which it could be used in a lesson.
>
> To represent functions or geometric figures, it is customary to use two perpendicular axes, x and y, which intersect at a point called the origin. These are known as the Cartesian axes, in recognition of the French philosopher and mathematician René Descartes (1596–1650), who was the first to employ them. If you were to plot the function $f(x) = -3x + 4$, defined on R, what would its graph be?

Figure 9. Task prompt proposed by MTE3 (Da Costa & Scorza, 2012, p. 42).

This task was initially considered to fall under Theme 5 because the construction of the object to be taught envisaged by the MTE was conceived with a mathematics lesson in mind: "Worthwhile teacher education tasks are those that are motivated by desire to foster the orientation in prospective teachers to the study of practice" (Zaslavsky & Sullivan, 2011, p. 7). However, the task did not entail analysing real or simulated practice, nor did it involve lesson planning; rather, the aim of MTE3 was to develop a concept map of the linear function (Figure 10) in dialogue with the PMTs.

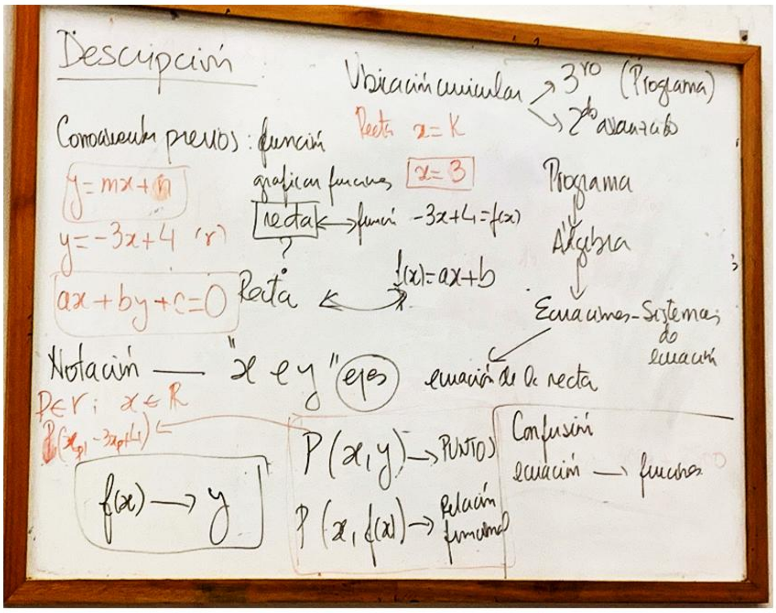


Figure 10. Image of MTE3's board.

Elaborating the concept map entails weaving together the relations within the topic of functions and the associated concepts and procedures. It is not simply a matter of recalling the definition of a function or its representational registers; rather, referring to the linear function will make connections to the equation obtained when finding its roots, relate the function to the equation of the straight line and to systems of linear equations, and link the function to Cartesian axes and hence to points given by their coordinates. Both the notation (x, y) and $(x, f(x))$ involve a specific distinction in the context of work with functions and their graphical representation. Likewise, thinking about the graphical representation means not only recognising that there will be an x-axis and a y-axis; for students, labelling them as x and y is not the same as labelling them x and $f(x)$, respectively. These examples—only part of what was discussed in the lesson—illustrate the complexity of constructing the object to be taught to which the MTE refers. The task arguably introduces a dimension broader than *Learning from the study of practice*, because at stake here is learning to study the construction of the object to be taught. Such construction is part of the work of a mathematics teacher and is undertaken prior to lesson planning and delivery. Consequently, this type of task, with the aims for which it was proposed, could constitute a new theme that extends the categorisation suggested by Zaslavsky and Sullivan (2011) and adds considerable richness to teacher education, as it requires envisioning the topic to be taught in terms of curriculum and mathematical connections (Appova & Taylor, 2019). These aspects have much in common with Theme 1 and Theme 2; however, in the case of this MTE, the purpose was different. In the post-observation interview, when asked how the task set was connected to the aim of teaching how to teach mathematics, the MTE responded: "For me, the ability to problematise mathematical content—the objects—is constitutive of knowing how to teach." [MTE3, I2] The MTE then added that:

> Learning to teach also means learning to design and to construct those objects. Construction involves mathematical analysis, but it also entails the specific didactic design that I am creating; there is a teaching project that is personal and contextualised, and it has to be built—it is not given anywhere, neither in the syllabus nor in mathematics textbooks. There are didactic materials that contribute to this, but classroom practice—the practice of teaching—has multiple sources; what I do with the learner requires different moments of creation, of intellectual creation. [MTE3, I2]

MTE3 positioned the teacher as an intellectual who constructs their teaching project and, to that end, must also construct the mathematical object to be taught in a distinctive way, since this is closely tied to their understanding of that object. This reflection—conceiving learning to teach as entailing the problematisation of mathematical content and the construction of a personal, situated, and creative teaching project—resonates with Giroux's (1988) view of teacher education as a space oriented towards educating critical intellectuals rather than technicians subordinate to curricular prescriptions. From within mathematics education, Gutiérrez (2018) offered a complementary perspective by arguing for the re-humanisation of mathematics, emphasising that teaching should be recognised as a situated and diverse practice in which creation occupies a central place. Although her emphasis falls on students' experience, her arguments can be extended to teacher education, reinforcing the idea that the prospective teacher needs to construct their own teaching project as an intellectual production.

Connecting Tasks to the Aim of Teaching How to Teach

In teaching how to teach mathematics, MTEs emphasized different aspects of practice and made explicit the intentions that guided their work. MTE1 indicated that her aim was for PMTs to understand that teaching is not merely a matter of setting a task but also of orchestrating its implementation and leveraging its potential in the lesson. For example, teaching how to teach entailed opening for discussion how a task is conceived before, during, and after its use. In the same vein of lesson orchestration, MTE8 sought for her students to envisage different ways of conducting the same activity, showing that the effects depend on its management and emphasising that students should play a leading role in the construction of knowledge. MTE10 proposed rehearsing and problematising the whole-class discussion, with the intention that PMTs understood that this phase should not be reduced to copying solutions on the board but should promote interactions in which ideas are debated and consensus is built. This is consistent with authors who argue that the impact of tasks resides not only in their content but also in how they are designed, implemented, and mediated in practice (Swan, 2011; Watson & Bills, 2011; Watson & Mason, 2007).

MTE2 sought for PMTs to appreciate the complexity of teaching through simulations and rehearsals of methodologies, so that they learnt to deal with contingencies and to connect classroom practice with research and with students' textbooks. There was also the intention of MTE3 that PMTs learnt to construct the object of teaching, recognising what is salient mathematically and didactically and making context-sensitive decisions. Other MTEs placed the emphasis on the analysis of teaching resources. The MTE4s aim was for PMTs to learn to appraise the mathematical and pedagogical potential of different materials, considering the representations and interactions they afford, while at the same time modelling a reflective planning process. In another case, MTE5, the task was oriented towards guiding PMTs in sourcing materials and in connecting content with other disciplines, with a view to informing lesson planning and task selection. Across these cases, the MTEs foregrounded the resources and materials teachers used to plan their teaching. The focus was primarily on Knowledge of Content and Teaching, though in MTE3's case it also involved Specialized Content Knowledge (Ball et al., 2008).

The cognitive demand of tasks was a recurring theme for two MTEs. MTE6 aimed for PMTs to learn to work with high-cognitive-demand tasks and to appraise what happens after group work—especially during the whole-class discussion and the closure—so that they understand how these phases affect learning. The other MTE (MTE11) stated that her intention was for PMTs to learn to enrich and render more complex classroom tasks, thereby increasing their cognitive demand, and to experience first-hand the demands of tackling more challenging tasks, so that they can transfer this experience to their practice. In this respect, the teaching aligned with recommendations in the literature that call for attention to the cognitive demand of tasks set for students (e.g., Smith & Stein, 1998). Student thinking emerged as a focal concern in another response by MTE7: teaching how to teach involved supporting PMTs in developing an interest in understanding students' thinking so that they can intervene more meaningfully in the classroom. The instructional focus was on Knowledge of Content and Students (Ball et al., 2008) and, more so, on fostering sensitivity to students' thinking (Jaworski, 1994; Runesson et al., 2011).

Teaching also involved rethinking proof and types of proof, with the intention that PMTs learnt to value different forms of argument and to question traditional practice that privileges fixed rules without discussion highlighted by MTE9. In a similar vein, MTE12 proposed analysing the language and discursive practices in use in the mathematics classroom, with the aim that PMTs learnt to recognise the degree of formality in statements and to choose formulations that support students' understanding and communication. In both cases, the goal was to foster critical reflection on the proofs brought into the classroom (Hanna, 1989) and on the use of different languages in lessons—such as gestural, symbolic, or graphical—consistent with the social and contextual characteristics of language (Moschkovich, 2007; Sfard, 2001).

In two cases (MTE3 & MTE11), the stance went beyond the mathematical and pedagogical aspects of mathematics teaching. For example, MTE3 emphasised that teaching how to teach involved fostering in PMTs the development of a personal teaching project that led them to decide what to teach and how to do so based on a situated, personal vision. As discussed earlier, this emphasis could be understood in dialogue with Giroux's (1988) notion of the transformative intellectual. From this standpoint, MTE3's proposal was not confined to transmitting Mathematical Knowledge for Teaching (Ball et al., 2008), but sought to support PMTs in developing a critical, personal perspective on teaching that enabled them to project their practice as a situated and transformative act. MTE11 stated that her objective was not only to teach how to teach mathematics, but to learn DM to reflect on the work of teaching; in her words, "teaching how to teach is teaching a way of thinking when one comes to plan a mathematics lesson." [MTE11, I1] That is, she invited PMTs to conceptualise teaching in its complexity, attending to content, interactions, context, educational level, and theoretical insights—consistent with Rojas et al. (2021).

These statements indicate that, beyond teaching theoretical aspects of DM linked to practice, some MTEs seek to enable PMTs to develop a complex and personal way of thinking about teaching—one that may guide their professional decision-making to go beyond the application of resources. The primary source for cultivating such a complex orientation lies in understanding DM as a disciplinary field.

Conclusions

With respect to the first research question, the dominant theme of each task implemented in the lessons observed were identified and assigned to one of the categories for secondary mathematics teacher education provided by Zaslavsky and Sullivan (2011). Under this criterion, the tasks were distributed across the following themes: *fostering awareness of similarities and differences*, *designing and solving problems for use in the mathematics classroom*, *learning from the study of practice*, *selecting and using (appropriate) tools and resources for teaching*, and *identifying and overcoming barriers to students' learning*.

There were no tasks whose specific objective was *developing adaptability*, *coping with conflicts and dilemmas*, or *sharing and revealing self, peer, and student dispositions*. Developing adaptability equips teachers to manage contingencies (Rowland et al., 2005). In line with Leikin (2021), preparing PMTs to confront conflicts and dilemmas will enable them to understand and respond more effectively to the demands of professional practice. Attention is warranted by sharing and revealing self, peer, and student dispositions, because PMTs' beliefs about mathematics and about its teaching and learning need to be made explicit to raise awareness of those beliefs and to contrast them with insights from research (Beswick, 2005). As noted above, these themes did not constitute specific foci of the tasks, but they were addressed tangentially to varying degrees. In general, the themes of the tasks addressed in the lessons observed fall within those recommended by Zaslavsky and Sullivan (2011) for mathematics teacher education.

Regarding the task used by MTE3, "Constructing the object to be taught and its dimensions", it does not fit neatly within the themes suggested by Zaslavsky and Sullivan (2011). A concept map of the topic to be taught contributes to the development of the complex mental map a teacher must construct when preparing to teach a topic; it is not reducible to lesson planning, activity selection, anticipating students' responses, or foreseeing interactions. It resembles the map metaphor used by Skemp (1976), where he contrasts knowing how to travel from one point to another in a city by carefully following a single route with having the city's map in mind to travel between points in multiple ways. When PMTs begin the work of thinking about the teaching of mathematics, they tend to focus almost exclusively on the lesson objective, the activities they will propose, the phases of the lesson, and the interactions to be fostered; but, in general, owing to their limited experience, their thinking does not yet encompass the concept map to which the MTE refers. Further research is needed to determine whether this theme warrants the creation of a new category.

With respect to the second research question, the analysis of the data indicates that MTEs employ a wide variety of tasks to connect their teaching with the aim of teaching PMTs how to teach mathematics. These tasks prompt PMTs to reflect on: task design and redesign (including, for example, raising cognitive demand); students' thinking; lesson planning; methodological orientations (ranging from more teacher-directed lessons to lessons centred on students' mathematical activity); lesson orchestration (whole-class discussion, patterns of interaction); appraisal of the potential of resources (manipulatives and technological tools, among others); types of proofs; and the languages that may be present in the mathematics classroom. Across cases, MTEs report their use of tasks as oriented towards helping PMTs grasp the complexity of organising teaching and learn to make well-founded decisions about what and how to teach mathematics (Ball et al., 2008; Escudero-Ávila et al., 2021; Rowland et al., 2005; Zaslavsky & Sullivan, 2011). In particular, the diversity of task foci accords with the findings of Escudero-Ávila et al. (2021) concerning the distinctiveness of the views held by MTEs who teach DM on how PMTs should be prepared. The purposes of these tasks are informed by theoretical knowledge in DM, by experience, and by MTEs' beliefs (Chapman, 2021; Chick & Beswick, 2018; Escudero-Ávila et al., 2021); the participants themselves made these sources explicit.

In a complementary vein, it appears MTEs also articulate educational purposes that transcend classroom-oriented concerns or the teaching of Mathematical Knowledge for Teaching (Ball et al., 2008) and are guided by political-educational stances. Some make explicit that they conceive their role as agents who support PMTs in constructing a personal educational project—at once situated and collective—and in developing a complex way of thinking about teaching, underpinned by the study of DM and the integration of content, interactions, contexts, and theoretical frameworks. From the outset of initial teacher education, PMTs are conceived as intellectuals in the making (Giroux, 1988). In this respect, the perspective adopted here—moving from tasks to formative purposes—enabled the research to make visible aspects of MTEs' teaching that are not fully captured by teacher-knowledge-based models.

Designing tasks that accommodate the wide diversity of formative purposes in mathematics teacher education poses a challenge for MTEs; moreover, resources for teaching DM are scarce (Chapman, 2021; Zaslavsky, 2008). In addition, "little support exists to help teacher educators develop their practice" (Ball et al., 2009, p. 459). The valuable knowledge, experience, and stances contributed by the participants in this study highlight the potential for MTEs to collaborate in the co-design of tasks and development of materials for the preparation of PMTs. The collective wisdom emerging from the practice of teaching DM in teacher education in Uruguay offers valuable insights that could inform and strengthen this work.

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Ethical approval

Institutional approval for the research was granted through the established procedures of Ochoviet's institution, Consejo de Formación en Educación, Uruguay. The study complied with relevant ethical standards, and informed consent was obtained from all participants for the use and publication of their data.

Competing interests

The authors declare there are no competing interests.

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