

Teacher Noticing: Attending To and Interpreting Students' Mathematical Fluency Through Written and Oral Representations

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Teacher noticing of students' thinking is a key component in the development of teaching expertise. The central focus of the study was on investigating teachers' noticing-attending and interpreting-of mathematical fluency. This article reports on findings from nine primary teachers' observations of upper primary students during mathematics lessons and their reported noticings in post-lesson group discussions. Teachers' noticings focused on attending to mathematically important details of students' strategies which aligns with previous teacher noticing research. The study contributes to the field of teacher noticing by illustrating how the use of a detailed observational tool provided specificity of the mathematical features to attend to, and a common language for discussion. The tool's emphasis on observing students' oral and written representations widens the scope of what evidence teachers used to interpret mathematical fluency (differing from how fluency is generally observed via numerical/written responses alone). In this study's case oral responses were most important; teachers relied heavily on restating student oral responses to interpret their noticings.

Keywords: teacher noticing and interpreting • mathematical fluency • student mathematical thinking

Introduction

Fluency in representations and procedures is foundational to primary students' development of mathematical proficiency. The *Australian Curriculum: Mathematics* (Australian Curriculum and Reporting Authority [ACARA] 2025) states:

Mathematics provides opportunities for students to develop, practise and consolidate skills; choose appropriate procedures; carry out procedures flexibly, accurately, efficiently and appropriately; and apply knowledge and understanding of concepts readily. Students are fluent when they connect their conceptual understanding to learned strategies and procedures, choose and use computational strategies efficiently; when they recognise robust ways of answering questions; when they choose appropriate representations and approximations; when they understand and regularly apply definitions, facts and theorems; and when they can manipulate mathematical objects, expressions, relations and equations to find solutions to problems. [Proficiency in mathematics, para. 3]

Mathematical fluency requires students to have access to and produce a range representations if they are to choose appropriate ones. The use of multiple representations allows students to "describe how they solved problems and why they purpose certain strategies and approaches" (Franke et al., 2009, p. 381) as part of sharing their mathematical thinking. According to the ACARA (2025) definition, students' demonstration of *mathematical* fluency extends beyond automaticity with facts alone, which is how *procedural* fluency is often interpreted. To observe students' mathematical fluency, teachers need to attend to the characteristics and affordances of representations. Cai et al., (2005) stated that students' representations "are the visible records generated by a solver to communicate her or his thinking" (p. 222). Representations afford teachers with viewpoints to observe fluency such as; orally (modality of language, see Cartwright, 2020) written (varying strategy-use see Jacobs et al., 2010) and



symbolic/diagrammatic (process understanding/misconception, procedural notation, accuracy, see Pimm, 2019). Therefore, in addition to the “traditional” focus on numerical/symbolic representations, attention should also be given to students’ oral and written descriptions.

Teacher noticing is a key component of teaching practice; what teachers notice can influence students’ learning and classroom experiences. There are two processes central to teacher noticing, attending to particular events or aspects of students’ mathematical activity, and making sense of students’ thinking within classroom interactions (Sherin et al., 2011). More recently, van Es and Sherin (2021) expanded teacher noticing to include *shaping* interactions to aid in creating space for observations of student thinking. In relation to Mason’s (2002) concept of the discipline of noticing, he noted that developing skills of intentional noticing is complex. Teachers need to know what to “selectively” notice and how to analyse what they observe when developing skills in noticing. Krupa et al. (2017) noted that “teachers must notice (attend to) the specific mathematical ideas evident in students’ written work or verbal responses in order to make sense of (interpret) that thinking” (p. 51).

Despite the substantial body of research on teacher noticing (König et al., 2022), research examining classroom-based teacher noticing of students’ mathematical fluency remains limited. The current study proposes to address the research gap, investigating the characteristics of students’ mathematical fluency teachers *attend to* as well as how teachers *interpret* what they notice. A focus on teacher noticing of students’ mathematical thinking aims to develop teachers’ capacity to notice characteristics of mathematical fluency communicated by students through a range of representations. This study explores teacher noticing through using an observational tool—*what they notice*, and analyses teachers’ interpretations of what they discover about students’ mathematical fluency—*how they notice*. The present study contributes to the teacher noticing literature by answering the following research questions:

1. *What do teachers attend to when noticing students’ mathematical fluency?*
2. *How do teachers interpret what they notice regarding students’ mathematical fluency?*

Findings related to these two research questions may also add to existing teacher noticing research by providing insights about two points raised by van Es (2011). First, van Es reflected that her teacher noticing framework did not “specify the kinds of student thinking that groups of teachers may analyse, such as student errors or solution strategies” (p. 149). Second, van Es proposed that future research should investigate “particular representations of student thinking” (p. 149). Focusing data analysis on student work samples and oral and written responses has the potential to reveal how students communicate their thinking and mathematical fluency through a range of representations, including, but extending beyond, traditional numerical/symbolic representations.

Literature Review

Teacher Noticing

A wealth of previous research exists on teacher noticing, dealing with prospective teachers including supporting preservice teachers’ noticing through: rehearsals (Anthony et al., 2015); focusing on mathematical thinking (Jacobs & Philipp, 2004); noticing mathematical strengths (Kalinec-Craig et al., 2021); using an analysis framework (Mitchell & Marin, 2015); self-contemplation to develop noticing skills (Sevinc & Galindo, 2022); and noticing in initial teacher education courses (Tekin-Sitrava et al., 2022). Research also investigated practicing teachers’ noticing such as elementary teachers’ expertise in noticing (Jacobs et al., 2024); how primary teachers select which students to notice (Jazby, 2016); primary teachers’ noticing during task design (Choy, 2016); elementary teachers’ noticing of mathematical reasoning (Melhuish et al., 2020); and teachers’ development of noticing skills (van Es, 2011). Much of the research concentrated on noticing teacher actions, investigating in-the-moment teaching decisions (Jazby, 2016; Sherin et al., 2011a; Sherin et al., 2011b) often utilising lesson observations during live lessons (Bragg et al., 2016; Melhuish et al., 2020) or observations of video-taped



lessons (Brantley-Dias et al., 2008; Sherin et al., 2009; van Es & Sherin, 2008). Several reflective reviews of teacher noticing on recent research developments (Weyers et al., 2024), a bibliometric review (Wei et al., 2023), and on analytic frameworks for teacher noticing (Amador & Weston, 2024) have also been published. Research on teacher noticing generally has the purpose of improving teachers' "professional eye" (Bragg et al., 2016, p. 524) for noticing important aspects of teaching and learning practices.

Teacher Noticing of Mathematical Thinking

Within teacher noticing research, student actions and mathematical thinking have been examined, for example: high school students' problem solving and reflection (Kooloos et al., 2022); the mathematical details in elementary students' strategies (Jacobs et al., 2010); elementary students' ability to justify and generalise during problem solving (Melhuish et al., 2020); and middle school students' novel strategies, error-making, and opportunities to share ideas (Sherin et al., 2009)—see Schack et al. (2017) and Sherin et al. (2011a) for other examples. Most of these studies focus more broadly on mathematical thinking and strategies across mathematical concepts. There are, however, some studies that deal with specific concepts: such as elementary students' early arithmetic strategies (Schack et al., 2013); noticing young students' thinking about the equals sign (van den Kieboom et al., 2017); and elementary students' algebraic thinking (Fisher et al., 2019).

Teacher Noticing of Mathematical Fluency

Teacher noticing of mathematical fluency involves attending to and interpreting students' "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately, and, *in addition to these procedures* [emphasis added], factual knowledge and concepts that come to mind readily" (Watson & Sullivan, 2008, p. 112). To notice mathematical fluency requires mathematical problems that have an element of challenge as these types of investigative tasks often take considerable time to solve (Russo & Hopkins, 2018). Students taking more time to solve problems increases the opportunities for teachers to observe mathematical fluency in action. Tasks with more than one solution or with a solution not immediately obvious are more likely to result in students producing multiple attempts at solutions and increase student to student discussion (Sullivan & Mornane, 2014). Research on mathematical fluency has predominantly focused on number concepts, operational procedures, and numerical facts, including arithmetic strategies such as addition and multiplication (e.g., Dole et al., 2018; Kamii et al., 2001; Russo & Hopkins, 2018). Similarly, classroom assessment of fluency has often emphasised students' proficiency with numerical and symbolic procedures. This focus highlights a need for research that examines how mathematical fluency is expressed and recognised across a broader range of mathematical activity and representations, including students' oral and written descriptions of their thinking.

Teacher Noticing: Scaffolds for Noticing Teacher Actions

Findings reported on within the teacher noticing body of research state that teacher noticing of student knowledge and skills (Brantley-Dias et al., 2008) requires a scaffold to guide the noticing process. Several studies developed frameworks for use during teacher noticing (Mitchell & Marin, 2015; Santagata et al., 2007; Tripp & Rich, 2012) and for researchers to analyse teacher noticing data (van Es, 2011; van Es & Sherin, 2002; 2008; 2021). van Es' (2011) framework of learning to notice captured the developmental phases of teacher noticing. van Es' study reported that novice "noticers" focused on whole class behaviors, participation, and student learning, compared with expert teachers who notice "particular students' mathematical thinking and the teacher's pedagogy" (p. 140). Sherin et al.'s (2009) framework focused on noticing evidence of student mathematical thinking within the lesson, answering questions like, "Are students exploring substantive mathematical ideas?" (p. 216). In these two studies (Sherin et al., 2009; van Es, 2011), the noticing of mathematical thinking is related to the

lesson itself, what opportunities the lesson poses for students to think mathematically and can teachers make the connection between teacher pedagogy and student thinking.

Teacher Noticing: Scaffolds for Noticing Student Thinking

Limited research exists that presents a framework related specifically to students' mathematical thinking for use by teachers in preparation for and during the noticing event (Choy, 2014). One framework for noticing teacher actions that includes aspects of noticing student thinking is the Mathematical Quality of Instruction (MQI) instrument designed by Beisiegel et al., (2018). While watching video-taped lessons, teachers in Beisiegel et al.'s study were guided by the instrument to notice student practices such as: students provide explanations, student mathematical questioning and reasoning, and students communication about the mathematics. Likewise, Melhuish et al. (2020) used the MQI to investigate teacher noticing of student thinking – mathematical knowledge (content) and the additional inclusion of mathematical reasoning (form). Melhuish et al.'s noticing of students' mathematical knowledge and mathematical reasoning align with the current study's focus on mathematical fluency. Students' ability to communicate both their knowledge and reasoning are important characteristics of mathematical fluency. Beisiegel et al.'s tool provides a starting point for noticing student mathematical thinking as one element of observing classroom practices.

Theoretical Framework

Research about teacher noticing has been ongoing for the past 25 years (e.g., Mason, 2002; van Es & Sherin, 2002). Over the past two decades, teacher noticing has become an increasingly prominent research approach for examining teacher actions and student thinking in classrooms (e.g., Goldsmith & Seago, 2011; Jacobs et al., 2010; Louie et al., 2021; Schack et al., 2017; Stockero 2021; van Es & Sherin, 2021). Several frameworks for teacher noticing in mathematics exist, most based on work by Jacobs and colleagues and van Es and Sherin (Amador & Weston, 2024). Teuscher et al.'s (2017) study utilised Jacobs et al.'s (2010) framework as a theoretical lens to view *attending* as the mathematical thinking teachers noticed and *interpreting* as how teachers made sense of the mathematical thinking to which they had attended. Part of Sherin et al.'s (2009) investigation of teachers' professional vision included the development of criteria to investigate whether a discussion of students' mathematical thinking was "productive", these were used as an analysis tool. The three criterion were utilised as codes to evidence noticing strategies used by teachers when discussing video-recorded classroom interactions: "understanding student thinking, substantive mathematical ideas, and joint sense making" (p. 221).

Both Jacobs et al.'s *attending* and *interpreting* framework for teacher noticing, and Sherin et al.'s (2009) coding descriptions are used within the present study. Jacobs et al.'s noticing framework was selected as it provides a construct to discuss further *what* teachers thought was evidence of mathematical fluency and *how* they justified their noticings. Sherin et al.'s strategy descriptions will be utilised during coding as they are useful when investigating how teachers' talk about their interpretations and what evidence they use in their justifications.

Methodology

Research Design

This qualitative research study, aimed at gaining insight in the form of rich narrative descriptions of mathematical fluency. The study's design had an emphasis on practitioner research (Cochran-Smith & Lytle, 2015) within the discipline of teacher noticing (Mason, 2002). To foster relationships between the researcher and classroom teachers a research design based within the classroom was selected. Classrooms provided an authentic context to build teacher skills in professional noticing of students' mathematical thinking (Jacobs et al., 2010) and provide space to foster deep intellectual discourse,



where there is a shared sense of “the practitioner as knower and agent for educational and social change” (Cochran-Smith & Lytle, 2015, p. 37). To illustrate the progression of the research, the design process is presented in Figure 1 along with the data collection points.

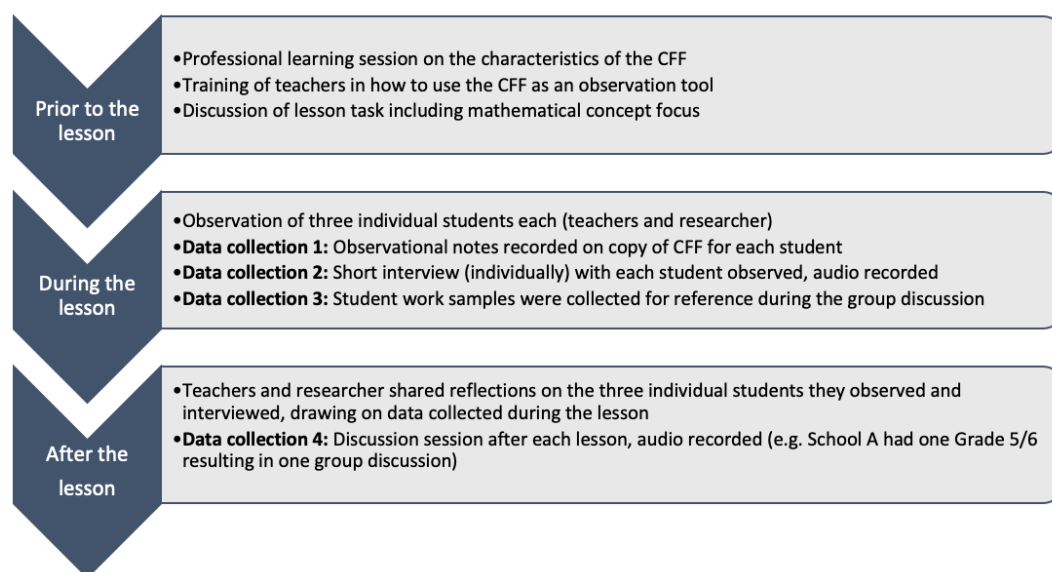


Figure 1. Research process and data collection points.

Study Background and Context

The results reported within this article form one component of a larger research study on the characteristics of mathematical fluency displayed by students, and what teachers notice regarding mathematical fluency. While previous research (Cartwright, 2018; 2019; 2020) reported on findings pertaining to the *student* data mapped to the CFF, this article focuses on the students through the *teachers'* eyes and the teacher noticing discussions that centred on data from mathematics lessons in four upper primary classes (Grade 5 and Grade 6 students, aged 10–12 years) from three schools involved in the study. *Data collection 4: Discussion session*, conducted after the lesson (see Figure 1), is the main focus of this article. The post-lesson discussions adopted a retrospective noticing approach (Sherin et al., 2011b). The purpose of choosing to discuss teachers' noticing of students after, not during the lesson, was to allow extended time for conversations with the teachers. Sherin et al. (2011b) reflected that “teachers may find reflecting on classroom activity and articulating what they notice easier in this retrospective-recall situation because they do not have to respond immediately to what they notice” (p. 82).

Researcher Positionality

Within the broader research study from which this article draws data, the researcher was an active participant in the delivery of the lessons across all classrooms. The lessons were taught by the researcher for consistency of delivery and to allow teachers to concentrate on the task of noticing and recording observations in relation to students' mathematical thinking. The researcher, a trained primary school teacher, taught the same 60-minute lesson in each classroom. The researcher observed and interviewed three individual students per lesson (like the role of a teacher). These data served as a shared reference point for the post-lesson discussions and provided the researcher with contextual knowledge to ask informed, relevant follow-up questions. In addition, the researcher facilitated the discussions with the teachers to focus teachers' noticing on identifying and interpreting student

responses for “evidence of students’ mathematical reasoning” (Goldsmith & Seago, 2011, p. 170) as part of mathematical fluency.

Participants

Nine primary school teachers participated in the study from three New South Wales (NSW) government schools. School A is a medium-sized school (high Non-English-Speaking Background (NESB), low socio-economic status context), School B is a large school with over 1200 students (high NESB, mid-high socio-economic status context), and School C is a medium-sized school (low NESB, medium socio-economic status context). These schools were selected purposively as they were representative of a number of differing school contexts within NSW, such as size, location, and level of NESB student populations. Teachers from these schools volunteered to participate in the study, which involved participating in teacher noticing events in their own and other teachers’ classrooms in their own school. For example, in School B, the Grade 5 teacher would observe students during the lesson in their own room, as well as students in the Grade 3 classroom lesson. For this article, the four Grade 5 and 6 classroom observations were examined: School A one class, School B one class, and School C two classes. Classroom teachers chose students from their own class (who had parental/guardian consent for participation) to record observations of during the lesson ($n = 3$ students per observer, 33 students observed, across the four upper primary classes). Teachers selected students from a range of perceived capabilities including students the teachers’ believed were representative of a mix of academic abilities. Figure 2 shows the observers (teachers), the classes they observed, and the number of students observed during each lesson.

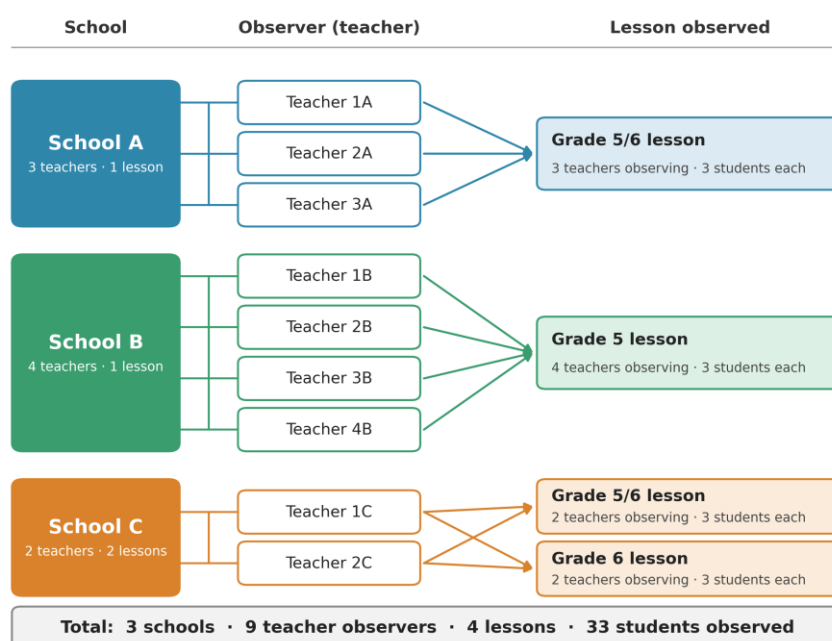


Figure 2. Participating observers and student numbers in upper primary classes

Teachers took on multiple roles during the research: observers/note takers of students during the lesson, interviewers of students towards the end of the lesson, and interviewees during post-lesson reflection discussions with the researcher. Teachers were assigned individual alpha/numerical codes to maintain anonymity and ensure confidentiality, for example, *Teacher 1A* is from School A. As well, pseudonyms are used when referring to individual students comments, responses, and work.

Study Procedures

Prior to the lesson: Professional development

Most studies focusing on teacher noticing mention the need for teacher professional development (PD) prior to conducting investigations to support teachers in their noticing (Bragg et al., 2016; Jacobs et al., 2010; Meluish et al., 2020; Mitchell & Marin, 2015). PD assists teachers in “what to look for” (van Es & Sherin, 2002) and how to make informed decisions (Jacobs et al., 2010) about “where to next” for students. Provision was made to ensure teachers were equipped with the knowledge to “notice particular things” (Bragg et al., 2016, p. 528) in the development of more accurate and purposeful noticing. To equip teachers, a 60-minute PD session was led by the researcher on the day of the lesson prior to its implementation. During the PD the researcher shared the purpose of the study, and the Characteristics of Fluency Framework (CFF) that would be utilised as an observation scaffold (see *Data Collection Instruments* section for elaborations). Each fluency characteristics was defined, together with descriptions of how it might be demonstrated in students’ responses. The researcher also addressed questions the teachers had about the CFF and clarified how it could be applied in practice.

The lesson to be observed was then discussed, particularly the mathematical concept the main problem-solving task focused on, along with possible student responses. The teachers and researcher did not trial the lesson beforehand, however, the researcher had used this lesson in a different school as part of the wider study. The PD session was conducted to make the participants familiar with certain noticing criteria from the CFF. The identification of a narrow focus for noticing (mathematical fluency) was necessary as teachers may “see classrooms through different lenses depending on their experiences, educational philosophies and backgrounds” (Jacobs et al., 2010, p. 171). Provision of a framework supported teachers in “watching, discussing, and reflecting upon” (Mitchell & Marin, 2015, p. 553) student data. The observing teachers were provided with the CFF recording sheet (see Figure 4 in the *Data Collection Instruments* section) to collect data, making observational notes on students. The observational notes were later drawn upon by teachers during the post-lesson group discussions to provide evidence of characteristics noticed in student responses.

The lesson structure

The lesson followed a launch, explore, summarise model (Lampert, 2001, more recently adopted by Sullivan et al., 2016). During the lesson’s launch phase, the role of the teacher was to be a silent observer as the researcher conducted an introductory whole-class number task focused on mental computation ($14 + 16 + 17 + 13 =$). The intention of the launch task was to act as a “warm up” for the lesson and to tune the students into the nature of thinking needed to engage them in deeper mathematical reasoning for the main task. Students then individually explored the problem-solving task depicted in Figure 3, which was adapted from Sullivan and Lilburn (1997).

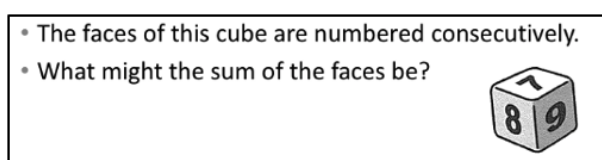


Figure 3. Problem-solving task: The dice question.

The problem has four distinct solutions (7, 8, 9 are **bolded** to show the progressive pattern):

1. 4, 5, 6, **7, 8, 9**
2. 5, 6, **7, 8, 9**, 10
3. 6, **7, 8, 9**, 10, 11
4. **7, 8, 9**, 10, 11, 12

Students were asked to solve the task and encouraged to write about *what they did* and *what they found out*. The researcher then led the whole class through the summarise phase of the lesson where students discussed their solutions and strategies. Questions posed gathered procedural data (*what did you do?*) and conceptual data (*what did you find out?*).

Data Collection Instruments

An Observation Scaffold for Teacher Noticing

Teacher noticing frameworks provide structures for directing teachers' attention towards particular aspects of classroom interactions and supporting their interpretation of students' mathematical thinking. However, the focus of noticing varies across research contexts, creating a need for frameworks that are responsive to the phenomena being investigated. Amador and Weston (2024) highlighted this need, arguing that "as the focus of teacher noticing evolves, so too should the mechanisms by which we assess noticing" (p. 1755). The MQI framework (Beisiegel et al., 2018) provides a broad structure for examining the quality of mathematics instruction and students' mathematical thinking. However, noticing specific characteristics of mathematical fluency requires a more fine-grained focus on how fluency is expressed in students' mathematical activity. The Characteristics of Fluency Framework [CFF] (Cartwright, 2018), Figure 4, provides a means of identifying and describing these characteristics, addressing the need to draw attention to key mathematical ideas (Choy, 2014; Krupa et al., 2017). It also presents a language for describing mathematical fluency, which has the potential to support more systematic observation and discussion of how students demonstrate fluency through their mathematical thinking and classroom interaction.

	Characteristics	V	W	Comments
Strategic competence	Appropriate strategy (for problem) Efficient strategy (for stage of learning) Variety of strategies Accurate process (correct solution) Multiple solutions Ease of mechanics (fluidity)* Flexibility (to switch between strategies)*			
Conceptual understanding	Making connections between concepts (known to unknown) Understanding and use of numbers and their relationships Explanation of method (process – how they worked it out) Description of thinking (what they found out) Sharing strategies – communicate/ collaborate with peers*			
Adaptive Reasoning	Justifying strategy (why the strategy worked) Justify findings (why answer is correct) Locates pattern in solutions / generalises Self-checking method (reasonableness -e.g. evidence of crossing out) Working through errors (evidence of further attempts once error identified)			
Representations	Use of a diagram - Picture - Lines or boxes Use of colour (to denote different solutions or attempts) Use of numbers (numerical representations) Use of symbolic representations Use of high modality words (if/then, always, so, therefore ...)*			

* generally verbal

Figure 4. CFF as observation recording sheet.

The CFF was developed in a previous study by the author (Cartwright, 2018), in which teachers were asked to identify characteristics of mathematical fluency displayed by their students. The characteristics identified included procedural competence and mathematical knowledge that aligned with existing research and definitions of fluency (ACARA, 2025; Baroody, 2006; Kilpatrick et al., 2001; National Council of Teachers of Mathematics, 2014; Morano et al., 2020; Zamarian et al., 2007). Teachers also identified characteristics such as giving reasons and explanations about knowing when and how to use appropriate skills. These responses extended beyond the characteristics represented in some existing definitions of fluency. Grouping the characteristics according to Kilpatrick et al.'s (2001) strands of mathematical proficiency assisted in distinguishing the *what* (understanding), the *how* (strategy), and the *why* (reasoning) of students' mathematical fluency. The CFF also incorporated the representational modes through which students communicated their mathematical fluency (Planas & Schütte, 2018), distinguishing verbal responses (V) from writing and drawn responses (W) (were also included (see Figure 4). Utilising a scaffold such as the CFF to guide teacher noticing directs attention towards students' activity, complementing approaches to noticing that focus on teachers' actions (Mason, 2002) to students' mathematical thinking (Jacobs et al., 2010).

Data Collection

During the explore phase of the lesson: Observation notes, student work samples and interviews

In each classroom, the teacher and the researcher observed the three designated students during the lesson. Each observer's three students were seated purposely at the same table. Observers recorded notes on the CFF recording sheet as part of data collection for each student, noticing how students were solving the task. Student interviews occurred near the end of the explore phase of the problem-solving process. Observers interviewed each of their three students individually while students had their work sample visible for reference (work samples completed during the lesson were also collected and analysed). All interviews were audio recorded. The interviews focused on students discussing their thinking verbally once they had worked on the task. Questions posed during the interviews included: *What did you do? What did you find out? What strategies did you use? What did you do to solve the problem and why?* Interview recordings were transcribed by the researcher for analysis. In-depth analysis of these interviews is not explicitly reported within this article; however, teachers often drew on the interview data as a useful catalyst for discussion (Jacobs & Philipp, 2004) during the post-lesson discussions.

Post-lesson: Discussion

Discussions, as a form of informal conversational interview (Patton, 2015), were conducted after each lesson. The observing teachers and researcher participated in the discussions. Four discussions were held across the three schools after each upper primary lessons: one in School A, one in School B and two in School C (see Figure 2 in the *Participants* section). Teachers only attended discussions at their own school. The discussions were unstructured conversations aimed at gathering teachers' perspectives and 'noticings'. Each post-lesson discussion continued for approximately 30 minutes and was audio recorded and later transcribed by the researcher for analysis. The purpose of the discussion was to reveal what the teachers *attended* to about the mathematical fluency of the students they observed and how they *interpreted* these noticings. The CFF directed the conversation by providing teachers with a scaffold to analyse the student data, which included classroom interactions and interview responses. Student work samples were also visible during the discussions; observers used these data sources to confirm and justify their comments. Clarifying questions were posed by the researcher during the conversations to discover: if teachers prompted students and what, if any, representational modes students accessed. Teachers' noticing of characteristics of mathematical fluency and their interpretations regarding students' fluency, were a rich source of data.

Data Analysis

The post-lesson discussion data were analysed for two purposes: to ascertain what teachers noticed regarding students' mathematical fluency, and to identify the evidence teachers drew on (Jacobs et al., 2010) to make their interpretations of what they noticed. Lincoln and Guba (1985) emphasised the relational nature of qualitative research and suggested checking back with participants to ascertain, "Have I got it right?" when making claims and interpreting responses. Checking back occurred in the research during the discussions where teachers were able to justify their interpretations verbally when questioned about what they noticed in students' responses.

Deductive and inductive analysis approaches

A combination of inductive and deductive analysis processes (Fereday & Muir-Cochrane, 2006) were employed to make sense of the teacher and researcher discussion data. Deductive analysis sets out to test consistency with prior assumptions (Thomas, 2006). In the case of the present study, deductive analysis assisted in answering the first research question: *What do teachers attend to when noticing students' mathematical fluency?* The CFF was utilised as a pre-defined code list for deductive analysis of the discussion transcripts to determine *what* teachers noticed. Prior to analysing the discussion data, the researcher mapped the student interview data and written data (work samples) to the CFF. The teacher discussion data, together with the notes from the CFF were then mapped alongside the researcher's analysis of individual student observation data and the observer's interpretations reported in the researcher and teacher discussions (Table 1).

Table 1.

Excerpt from the Mapping of a Student's Data

CFF Characteristic/s	Student responses	Researcher comment	Teacher noticing
Multiple solutions Use of numbers and their relationships Justifying findings (why the answer is correct)	I would add the additional numbers together and then take the answer that I got out of that answer and then if I got 24 that would be correct [oral]. It would be correct because 24 is the sum of 7, 8 and 9 [written]	Good technique with working out the three numbers already visible on the dice make 24 then using that knowledge to work out the totals.	He had three solutions, and then, said to me, I don't need to work out the next solution, I can just add six to the total that I've got, and I know the next one. He could justify ... [Teacher 1C].

This initial analysis process assisted in checking the reliability of the researcher's and teacher's interpretations of the student data. The mapping process also provided affordances to collect evidence of how teacher noticing aligned to Jacobs et al.'s (2010) noticing skills: noticing descriptions including mathematical important details, and interpretations that specifically addressed the students' understandings. Inductive analysis (Thomas, 2006) occurred when the analysis of teacher discussion data moved from analysing what teachers were attending to about students' mathematical fluency, to answer the second research question: *How do teachers interpret what they notice regarding students' mathematical fluency?* Each discussion was analysed looking for teachers' interpretations—how they noticed students' mathematical fluency—searching for what evidence were they using. A second cycle of inductive analysis occurred where the transcripts were revisited in a broader sense with the primary goal of searching for emerging patterns in teachers' interpretations that could then be compared with Jacobs et al.'s (2010) evidence of robust interpreting (excerpt of data, Figure 5).

Interview extracts	Patterns within Jacobs et al.'s (2010) evidence of interpreting
Teacher 2A: Yeah, and to be able to reason on why something isn't working or why it is working and <u>being confident in being</u> - well that's how we would know students are fluent, by their <u>confidence</u> I'd say. Teacher 3A: Because in all - both classes the <u>variety</u> was so wide...	• Confidence in knowing why • Teacher comparison of students – level of fluency
Teacher 2B: Yeah, I think so. Most of <u>them</u> - a lot of <u>them</u> <u>were confident</u> that they had the correct answer; they were <u>quite confident</u> ... Teacher 3B: And that they found the <u>quickest and easiest way to do it</u> ...	• Confidence in correct answer • Efficiency
Teacher 2C: she <u>quite quickly worked out</u> there were four solutions, could justify, well, <u>there can't be any more than these four solutions</u> ... and she was the only one, <u>even though Polly was beside her Polly went top and tail, Lilly went top and tail</u> . So, I think she had a <u>better visual</u> of it than the other girls.	• Efficiency • Teacher comparison of students – level of fluency

Figure 5. Excerpt of patterns in teacher interpreting comments.

Results

Findings from the four post-lesson discussions are reported here collectively to explore what teachers attended to and how they interpreted what they noticed. Although all four focus discussions were collected separately, they were coded under the same analytical framework and are investigating the same phenomena. The datasets are therefore presented together to demonstrate complementarity and triangulation of findings related to the two research questions.

RQ 1: What do teachers attend to when noticing students' mathematical fluency?

The results from analysing the data for evidence of teacher noticing of mathematical fluency are discussed here organised by Jacobs et al.'s (2010) suggested teacher noticing skills that require attention. These noticing skills are (1) descriptions that include the mathematically important details of students' strategies, and (2) comments specifically addressing the students' understandings. An additional category of noticing students' verbal- and written-strategy explanations was included. Noticing verbal and written representations in Jacobs et al.'s study is linked to how teachers respond to children. To defend its inclusion here, the present study acknowledges that to respond to students' representations, you also need to notice them. Two example illustrations of teachers' holistic attending to students' mathematical fluency are then presented.

Attending to mathematically important details of students' strategies

Attending to mathematically important details within the data recognising "mathematically significant indicators in children's messy, and often incomplete, strategy explanations" (Jacobs et al., 2010, p. 194). Teachers identified evidence of students' mathematical content knowledge embedded in the students' actions, written notations and verbal responses. Comments by Teacher 3A are an example of how teachers described mathematically important ideas:

Jonah and Hayley understood consecutive, they understood sum, they understood solutions and they needed to be challenged further because they knew — they wanted to know why. Because when I asked them, "Is there a pattern?" they thought "Oh, what about if I start at the smallest one and go to the largest one" — and that's how they worked it out straight off — you just add six. [Teacher 3A]

Teachers in the discussion noticed characteristics related to students' strategy explanations, for example:

When I was asking Asa how she came up with all of her different combinations, she was saying that she had written seven, eight, nine and then explained that there were three sets because she could go on either side so she just made sure that if she had seven, eight, nine in the beginning that she'd come

up with what was after. If she had seven, eight, nine in the end, she'd come up with what was before.
Or if she had it in the middle, she'd come up with what was before and after. [Teacher 1A]

Teachers used the names of the characteristics from the CFF when they attended to students' strategies, for example, Teacher 4B said, "I know several of them used a variety of strategies," and Teacher 1A reflected that: "There was one student I had spoken to when I was talking [about finding] multiple solutions. She asked, 'Well, besides just addition, can I use multiplication?'"

Comments addressing students' understandings

Discussions based on understanding (content) were not as prevalent as those regarding students' strategies (process). Teachers' comments that specifically did address student understanding were rich with connections among students' strategies, their application of known facts, and/or transference of knowledge from one context to another:

With Evan, he was saying he used known facts and then an algorithm, so he would add as many numbers as he could as known facts, so he knew nine and eight made 17, he knew seven and six made 13... that shows me that he has that flexibility. [Teacher 1C]

She was like, "I just realised they all do that, she said each number is divisible by three so then you can add three numbers together." [Teacher 1C]

He said, "Another thing I've noticed is that seven plus eight plus nine equals 24 and that kept coming up." [Teacher 2C]

Noticing students' oral and written-strategy explanations

Teachers noticed that when students did not write sentences to support their thinking, it made analysing their mathematical fluency problematic. Teacher 1B reflected on the importance of students' written explanations by sharing insights from one of her Grade 5 students, Malak, "I heard him say, you need to show your working out ... it's because you can see where you've made a mistake with your working out." Teacher 1C also commented on the need for explanation, stating; "especially if they haven't written that down, how they did it, because then you just don't have anything to go off, you're like, how did you do that?" The importance of oral and written representations can be seen in the comments teachers made when noticing the presence, or absence, of explanations, for example:

He got the answer quite quickly and when I asked him to explain how he got the answer ... he couldn't actually articulate what he was trying to say but then when I said try and write for me – try and explain it to me on paper how you got that answer, he counted by ones. [Teacher 1A]

Yeah, he was also able to really well verbalise...what his thinking was. When he did this, he was then able to really put it into consistent language ... he was able to really put his thinking into words as well. [Teacher 4B]

If you don't have that verbal explanation ... he literally could have just sat beside somebody who had done it. [Teacher 1C]

He didn't write as much. He had a lot of numbers written down but not a lot of words. [Teacher 4B]

Example illustrations of teacher attending to mathematical fluency

The following two student work samples and teacher comments from the discussions illustrate how teachers noticed mathematical fluency in student responses. Duha's work sample from School A Grade 5/6 is presented in Figure 6 and includes letter codes, for example, Point A, that align to characteristics and teacher noticing skills in Table 2.



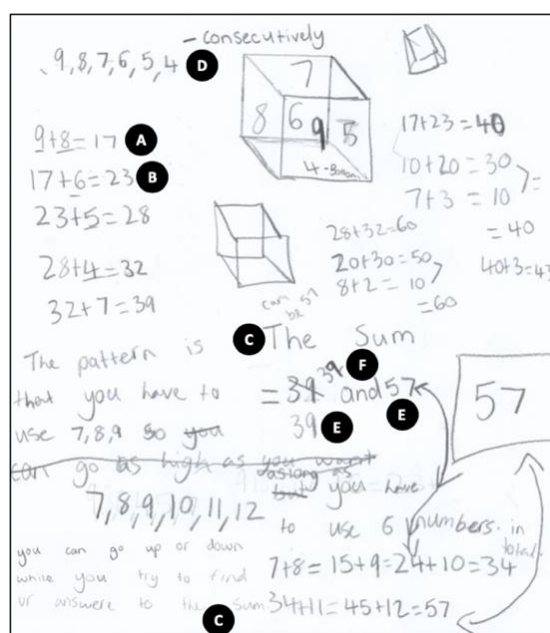


Figure 6. Duha's work sample [School A, Grade 5/6].

Table 2.

Teacher 1A's noticing of Duha's Mathematical Fluency

Letter code	Characteristics noticed by the teacher	Teacher comment	Jacobs et al. (2010) noticing skills
A	Justifying strategies	Duha was able to say, well I know nine plus nine equals 18 and then I just took away one	Attending to the details of children's strategies
B	Efficient strategies	She said, 17 plus three equals 20, 20 plus three equals 23 so that was good	Noticing children's oral- and written-strategy explanations
C & D	Comprehension of the task Understanding and use of numbers	I saw that they understood the word consecutively because they started writing consecutive numbers. I remember that one of them [Duha] was going seven, eight, nine, 10, 11, 12 and I was like okay, that's great ... once they understood what sum meant then they were able to at least try and complete the task.	Attending to the details of children's strategies
E	Multiple solutions	Duha was able to give two possible solutions.	Descriptions that include the mathematically important details
F	Self-correcting	Duha showed me her working out and she had 32. When I was asking her to recheck her numbers, she's like I've got nine, I've got eight, I've got six, five, four and then she had the answer as 39 but in her working out it said 32. I'm like why is there a difference there? Why have you got a different set of working out compared to your answer? She went through her numbers again and she's like oh, I missed the seven.	Comments specifically addressing the children's understandings

Chloe's work sample from School B Grade 5 is presented in Figure 7 and includes letter codes. For example, Point A, that align to characteristics and teacher noticing skills listed in Table 3.

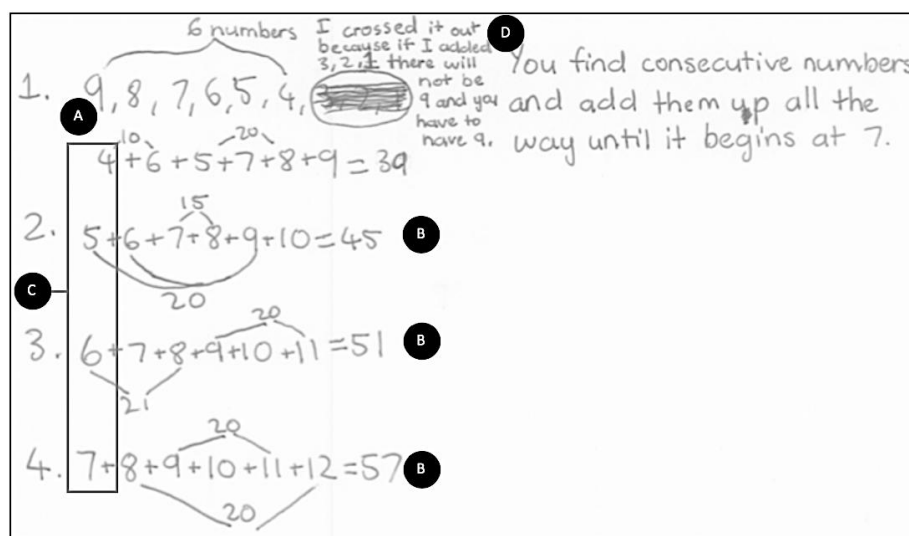


Figure 7. Chloe's work sample [School B, Grade 5].

Table 3.
Teacher 3B's noticing of Chloe's mathematical fluency characteristics

Letter code	Characteristics noticed by the teacher	Teacher comment	Jacobs et al. (2010) noticing skills
A	Efficient strategies	She started off with the four, five, six, seven, eight, nine and actually writing the numbers in that order - or backwards from that order, and then did the equation next to show how she matched the tens first and then added it that way.	Attending to the details of children's strategies Noticing children's oral- and written-strategy explanations
B & C	Multiple solutions Location of a pattern	She just went straight into the equations and was able to put them in the same order and then find that the pattern goes up by six.	Comments specifically addressing the children's understandings Attending to the details of children's strategies
D	Self-checking method Working through errors Justifying findings	I asked her why she crossed out three, two, one. She goes, well, I put that in to begin with, then I realised if I used three, two, one, I then can't use seven, eight, nine, so I had to cross that one out that didn't make sense; it wasn't part of the consecutive numbers in the dice that I had to use.	Comments specifically addressing the children's understandings Noticing children's oral- and written-strategy explanations

These two annotated work samples exemplify the complexity of teacher noticing skills and the intertwined nature of mathematical fluency. Many crossovers appear between students' strategies, understanding and reasoning characteristics that the teachers drew on to discuss each students' mathematical fluency. For example, when discussing Duha, Teacher 1A noticed that Duha could justify her strategy orally by using known facts. This one data piece is an example of the teacher displaying noticing skills that encompass both an ability to attend to the details of children's strategies and to notice children's oral-strategy explanations. Similar connections existed in Teacher 3B's noticing of



Chloe's mathematical fluency. Teacher 3B did not specifically describe the mathematically important details of Chloe's work, but did attend to the details of children's strategies, noticing how the student wrote the equations that later assisted the student in locating a pattern.

RQ 2: How do teachers interpret what they notice regarding students' mathematical fluency?

Jacobs et al. (2010) described levels of teacher interpretation – robust, limited, and lack of evidence:

To interpret children's understandings, one must not only attend to children's strategies but also have sufficient understanding of the mathematical landscape to connect how those strategies reflect understanding of mathematical concepts. (p. 28)

It was Jacobs et al.'s highest level of evidence—robust—that provided a way to organise teacher interpretations that aligned to the themes observed in the data. Most participating teachers acknowledged mathematics as their subject strength and were experienced teachers, it is therefore not surprising that these teachers were able to provide robust evidence of interpretations. The current study did not seek to compare the degree to which teachers interpreted, but simply to explore the interpretations themselves in relation to mathematical fluency. This section is organised by adapting the 'robust evidence' of interpretation Jacobs et al. proposed; (1) making sense of student strategy details in a variety of ways, (2) noting how these details reflect what the student did [or did not] understand—their confidence in knowing why, and (3) comparing strategies of students.

Making sense of student strategies

Teachers' interpretations about students' strategies were generally comparative in nature, students 'had it' or 'did not have it'. Commonalities appeared in the teacher comments when discussing how they interpreted students' strategies, understanding and reasoning: efficiency/inefficiency, and levels of fluency.

Efficiency/inefficiency. Both a presence and an absence of efficient strategies was noticed by teachers in making sense of student strategies. Teacher 1C mentioned that Grade 5/6 student, Polly, "was still quite quick" in reference to her fluidity in finding solutions. Whereas an absence of efficient strategies was noticed when Teacher 1C described Grade 5/6 student Laura's use of expanded notation to add her numbers together stating; "Her strategy was a lot slower ... Her strategy took quite a long time for her to get to an answer."

Levels of fluency. The idea of the existence of 'levels' of fluency was observed in several teacher comments, for example, "I really felt like Carly was working at a level above." [Teacher 1C] When discussing Carly, the teacher interpreted the student's competence in explaining what she was doing as an indication of 'higher' fluency. Teachers noticed 'well-developed levels' of fluency, such as Teacher 1C who when talking about Grade 6 noted, "I just thought, wow, that's really developed. I didn't see any other kids do that today". Whereas students who struggled to comprehend the task "showed fluency slightly." [Teacher 1A]

Confidence in knowing why

Two patterns emerged in teachers' interpretations. These patterns reflected the perceived reasons why students did (or did not) understand: language capability, and confidence.

Language capability. Students' comprehension of the task was interpreted by teachers as being influenced by understanding of key language features of the task, for example, "Once they understood what 'sum' meant then they were able to at least try and complete what the task was asking them to do" [School A]. When students "got stuck" [School A] with a word like 'solution', noticing characteristics of mathematical fluency was difficult without prompting. Teachers noticed that when students "struggled with the wording of it, the language 'the sum of those numbers'" [School C], it made choosing an appropriate strategy challenging. Conversely, teachers interpreted the use of high modality words as indicators of high levels of mathematical fluency, for example, "a lot of them also



used those modality words ... like, 'therefore'... and 'thus'" [School B]. An ability to use language features such as high modality words to communicate mathematical thinking was interpreted as a positive sign of mathematical fluency.

Confidence of response. Teachers correlated a student's level of mathematical fluency to confidence. Some teacher noticings aligned to Foster's (2016) definition as "confidence of response" (p. 274) by students in the correctness of their solutions; "he was quite confident with his answers" [School B]. Other noticings linked to confidence as mathematical self-efficacy (Bandura, 1977) within their own ability, "she was confident in herself with what she was doing" [School B], or confidence as perseverance, "that's that resilience to keep having that go" [School C]. Comments such as Teacher 2A's interpretation, "that's how we would know students are fluent, by their confidence I'd say," were observed in the discussions. For students who were unsure of their answers, or struggled to continue with the task, fewer characteristics of fluency were able to be noticed. Teachers suggested this may be a result of their lack of mathematics confidence (Pierce & Stacey, 2004), "I feel like they don't feel confident in maths, so they're not willing to keep trying." [School C]

Teacher comparison of students' strategies

For participating teachers, making comparisons between student strategies they observed was an essential element of interpreting student data. In the following example from School C, the teacher compared students based upon what she noticed students doing in relation to finding and justifying solutions and locating a pattern:

Lots of the kids saw two—top and bottom, but not the others. Whereas some kids can see the pattern of six but can't explain why six. [School C]

Comparisons were made between students to assist in illustrating specific characteristics such as reasoning, for example, Teacher 1C reflected on Grade 6 student Eva by stating "Eva said, *I think it's going up by sixes, because there are six sides on the dice.*" Teacher 2C then responded with, "compared to the kids like Tyson, who can explain, five and the one... *because I know why it's going up.*" At School A, two teachers compared students they observed justifying strategies when discussing if the Grade 5/6 students showed reasoning:

Teacher 2A: Jayden got the number really quick. He's got an understanding of I guess number and number sense but couldn't reason on how he did it in his head.

Teacher 1A: Whereas I had asked that question as well, and Duha was able to say *well I know nine plus nine equals 18 and then I just took away one* so that was good.

Teacher 2A stated the first student had "number sense but couldn't reason", then Teacher 1A used the term "whereas" to compare the second student who provided number knowledge as a reason for her strategy. Comparing students to describe who 'had' or 'did not have' a certain characteristic was also visible in other school discussions about reasoning.

Discussion

Results of the study indicated that with the guidance of a detailed observation tool, teachers attended to multiple fluency characteristics and displayed a variety of teacher noticing skills (Jacobs et al., 2010). Teachers focused on descriptions of mathematically important details of students' strategies, understandings and reasoning, as well as students' oral and written representations to discuss what they noticed.

Teacher Noticing of Student Mathematical Fluency

One of the most noteworthy findings from the current study was the in-depth way teachers' spoke about students' mathematical fluency. Teacher comments included descriptions of mathematical important details (Jacobs et al., 2010) related to students' thinking. Teachers focus on mathematical



detail was evident in the types of comments made in examining students' mathematical fluency such as strategic competence, "I know several of them used a variety of strategies," conceptual understanding, "with Evan, he was saying he used known facts and then an algorithm," or adaptive reasoning, "He [Tyson] was justifying, why it's six. Whereas some kids can see the pattern of six but can't explain why six." Teachers focus on the characteristics of mathematical fluency in their noticing provides the specific kinds of mathematical thinking van Es (2011) mentioned were not present in her framework. It is suggested that one potential reason for these results was the provision of the CFF as an observation tool accompanied by PD in the use of the framework prior to the lesson observations.

Teacher Interpreting of Student Mathematical Fluency

A second major finding was the similar patterns in the way teachers shared and reasoned about their interpretations. Results revealed common patterns within teachers' interpretations of what 'well-developed' levels of mathematical fluency looked like. Included were a student's use of oral- or written-strategy explanations, confidence of response, and a grasp of mathematical and general language features. Teachers' interpretations were often informed by the act of comparison. These patterns can be seen in Jacobs et al.'s (2010) robust evidence of interpretation aspects that the teacher noticing results were mapped against.

Teachers within the current study employed similar strategies to Sherin et al.'s (2009) participants when reasoning about student thinking (1) focus on student thinking *by restating student ideas* (2) mathematical substance of discussion *by investigating the meaning of student ideas*, or (3) joint sense making *by synthesising across the ideas of students* to make sense of what they noticed. The discussion of teachers' interpretations about students' mathematical fluency they noticed are discussed in the following section organised by these strategies.

Restating student ideas (quoting what a student said)

During discussions teachers frequently quoted students verbatim to provide evidence of students' mathematical fluency. One reason why the teachers may have restated student ideas was to illustrate a specific characteristic or to justify their thinking. Although teachers took turns in describing their observations during the discussions, teachers would find agreement in their noticings and interjected with a confirmatory quote from one of their own students. Another reason why these results may have been obtained was the inclusion of student interviews at the conclusion of each lesson, leading to teacher noticings based on students' oral responses, not just what they noticed by looking. The interviews shifted fluency observations from solely interpreting written numerical/symbolic work, to encompassing oral responses. This is a noteworthy new development in teacher noticing of students' mathematical fluency. Future research projects about teacher noticing of students' mathematical thinking may benefit from the use of student interviews to obtain the students' perspectives in addition to that of the teacher.

Investigate the meaning of student ideas

Students' representation modes allowed for the examination of the meaning of student ideas. The CFF encouraged teachers to notice the representations students displayed fluency through, particularly their written or oral responses. Within the present study, the exploration of students' various ways of representing their thinking provided the data van Es (2011) acknowledged should be the object of future investigation. Teachers also examined the meaning of student ideas in their noticings with an evaluative stance (Sherin & van Es, 2009) like "that was good," or an interpretive stance, "a lot of them were just looking for numbers that were easy to add together rather than a pattern". The evaluative stances were generally appended to an interpretive stance and were therefore viewed as higher level of noticing than evaluative comments alone.

Synthesising across student ideas

The evidence of interpretation ‘comparing strategies of students’ in the results highlighted moments when teachers synthesised data from multiple student responses to justify their thinking. Comparisons focused on the generalisation (finding examples and counterexamples) of fluency characteristics across student data, rather than a comparison of students’ abilities. This pattern of *synthesising across students* was observed in all four discussions. A reason for these findings may be that teachers knew the students. They were either students currently in their class, or students the teachers had previously taught. Teachers brought both their prior knowledge of students and student expectations to how they interpreted. Knowledge of this type provided a *contextual* lens (Brantley-Dias et al., 2008) for teachers to interpret student data through.

Importance of Confidence and Language for Teacher Noticing of Mathematical Fluency

An unexpected finding was teachers interpreting what they noticed as confidence of response. Confidence and its role in mathematical fluency is discussed by Foster (2016), who stated, “I consider a pupil’s ‘confidence of response’ as how certain they are that the answer they have given is correct” (p. 274). Foster differentiated confidence of response from other forms of social confidence: mathematics confidence (Pierce & Stacey, 2004), and mathematical self-efficacy (Bandura, 1977). Confidence in relation to students’ beliefs and feelings towards themselves as mathematics learners was one part of how teachers interpreted student confidence. This view of confidence is more akin to a need for students to develop mathematical resilience (Johnston-Wilder & Lee, 2010), a concept difficult to measure. Foster (2016) agreed, stating that confidence is difficult to assess reliably. Teachers within the present study, however, mentioned confidence numerous times, implying confidence of responses was a feature of students with well-developed levels of fluency. One teacher articulated the importance of confidence saying, “That’s how we would know students are fluent, by their confidence I’d say.” Although difficult to observe, confidence of response can be seen in students’ use of language features such as high modality words like *I knew* or *I know that*, and causal connectives such as *then*, *so*, *because*, and *therefore*, of which several were noted in the theme of language capability. In talking about fluency and how fluency is currently assessed, Foster (2016) reflected that “written mathematics assessments [of fluency] generally focus on competence with procedures, but an important concomitant of procedural competence is the pupil’s confidence in the answers that they give” (p. 272).

Limitations and Future Research

Limitations exist within the present study that need to be addressed. First, the study involved a small number of teachers ($N = 9$) across three schools, all of whom volunteered to participate in a mathematics research project. Future research that broadens the number of teachers involved in the study and includes teachers from diverse teaching backgrounds, beliefs, and experiences, may allow for further investigation of the usefulness of the CFF as a tool for teacher noticing. Second, the study was conducted based on a small set of lesson experiences of teacher noticing, a longitudinal study of teachers and their development of their “professional eye for noticing” (Bragg et al., 2016, p. 542) would be beneficial. Future research should continue to focus on mathematical fluency with the extension to teacher noticing of *teaching* so in-the-moment pedagogical and instructional decision making (Jacobs et al., 2010) can be explored. Studies of this nature will continue to inform educators and researchers about implementing strategies (teaching-focused) in the classroom that emphasise aspects of mathematical fluency, and the fluency characteristics students’ display in written and oral responses (student-focused).



Conclusion

The over-arching goal of the research was to investigate what characteristics of mathematical fluency teachers *attended to* in students oral and written representations and how teachers *interpreted* what they noticed. Teachers were able to identify multiple characteristics of mathematical fluency that aligned to Jacobs et al.'s (2010) noticing skills. Additionally, the investigation of students' oral *and* written representations to observe mathematical fluency is novel compared with previous fluency research and builds on Bay-Williams and Kling's (2019) research utilising interviews for students to share their reasoning. The gap in how students communicate fluency (oral and written), and how it is generally assessed (usually numerical only) has implications for future research. Attention to students' oral and written responses during noticing events may reveal evidence of students' mathematical fluency not able to be observed through written numerical work alone. Patterns were observed across the four discussions in how teachers interpreted their noticings providing reasons predominantly by restating student ideas. Quoting students to justify noticings is consistent with prior research by Sherin and van Es (2009) and van Es (2011). Teachers' interpretations also addressed factors that influenced students' "level" of mathematical fluency such as *confidence of response* and *language capability*. These areas warrant further research exploring the impact confidence and language acquisition have on students' ability to develop or communicate their mathematical fluency.

Findings from the current study contribute to the field of teacher noticing of student mathematical thinking illustrating Jacobs et al.'s (2010) noticing skills and providing evidence of robust interpretations of students' mathematical fluency. The combination of the CFF scaffold (Cartwright, 2018) with existing teacher noticing frameworks (Jacobs et al., 2010; Sherin & van Es, 2009; van Es, 2011) may assist in developing teacher's skills in noticing students' mathematical thinking. Scaffolds or frameworks assist in moving teachers from evaluative to interpretive observations (Sherin & van Es, 2009). For teachers to promote classroom practices that develop mathematical fluency in students, they need to know what to notice. Once teachers know what to look for and can recognise students who have not (yet) attained well-developed levels of fluency, "then teachers may look for [other] ways to address this issue in their classroom" (Melhuish et al., 2020, p. 37). The CFF is potentially an effective tool for intentional noticing (Mason, 2002)—specifically for identifying mathematical fluency. The employment the CFF to analyse student work samples, classroom lesson videos, or recorded responses in one-to-one or small group settings has the potential to benefit teachers in the development of their professional vision.

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Ethical approval

Ethical approval for the research was granted by the University of Sydney (HREC2018/810) and informed consent was given by all participants for their data to be published.



Competing interests

The authors declare there are no competing interests.

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