An Alternative Method for Division of Fractions: Situations of Contingency and Teachers’ Responses

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When approaching mathematical tasks, students occasionally introduce unconventional methods, which may be rejected by teachers despite their validity. This paper presents two studies conducted to examine teachers’ responses to an alternative method for division of fractions (AMDF), presented by a hypothetical student, in two different contexts. In Study 1, prospective teachers ($n = 36$) wrote about how they would respond to a student’s suggestion to use the AMDF. In Study 2, teachers with various levels of professional experience ($n = 6$) were interviewed to capture their responses to a student who uses both the AMDF and a standard division algorithm. The teachers’ responses from both studies were analysed by drawing on components of the Knowledge Quartet framework. The findings suggest additional refinement of the framework and illustrate a difference between teachers’ responses in writing and in an interview.

Keywords: student-invented strategy • teacher response • contingency • fraction division

Introduction

When students use diverse strategies to solve mathematical problems, their understanding of key concepts improves (Fennema et al., 1996; Jacobs et al., 2007). Teachers support learning by sharing those strategies with the class (Silver et al., 2005; Stein et al., 2008). Teachers, however, often struggle to interpret students’ unconventional methods (Jakobsen et al., 2014), sometimes attributing them to misunderstanding and steering students toward familiar approaches (Hines & McMahon, 2005). Responding in this way could not only discourage further student contributions but also limit students’ known strategies. It is thus important to understand how teachers engage with their students’ unexpected methods.

Teachers’ ability to accurately interpret student-invented strategies typically depends on their knowledge (Coskun et al., 2021), which is also affected by other complex factors. The context in which student thinking is presented influences teachers’ response to student contributions (Baldinger & Campbell, 2021). Although teachers’ expertise aligns with their ability to accurately assess student thinking during interviews with those students, this connection weakens when the teachers are only allowed to analyse the students’ written work (Magiera et al., 2013). Adequate knowledge also does not guarantee an effective response to an inventive strategy (Son & Crespo, 2009). Teachers’ responses can be affected by their perceptions of students’ mathematical proficiency (Wager, 2014). Our work extends this area of research by presenting two studies in which an unconventional method for dividing fractions is introduced as a student’s idea. Our goal was to explore how teachers engage with the unconventional method, in different contexts.

Fraction Division and an Unconventional Algorithm

For several reasons, fraction division stands out as one of the most problematic areas for mathematics learners. As a start, the result of division by a proper fraction is in discord with the naïve expectation that "division makes numbers smaller" (e.g., Graeber et al., 1989; Tirosh & Graeber, 1990). Further, learners often experience difficulty when asked to describe a situation that involves division by a fraction (e.g., Ball, 1990; Simon, 1993), as most people exhibit a partitive disposition towards the operation of division. Such a conceptualisation does not allow students to readily make sense of fractional divisors.

These difficulties are exacerbated when the standard algorithm for division—where division is replaced by multiplication and the divisor is replaced by its reciprocal—is introduced to learners without a proper explanation (Borko et al., 1992). There are, however, different strategies for justifying this procedure (see Son & Crespo, 2009, for a partial list). One such explanation is to follow a sequence of computations, such as:

$$\frac{2}{3} \div \frac{4}{5} = \frac{\frac{2}{3}}{\frac{4}{5}} = \frac{\frac{2}{3} \times \frac{5}{4}}{\frac{4}{5} \times \frac{5}{4}} = \frac{\frac{2}{3} \times \frac{5}{4}}{1} = \frac{2}{3} \times \frac{5}{4}$$

However, division of fractions can also be carried out by dividing numerators and denominators separately, that is,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$$

We refer to this strategy as the alternative method for division of fractions (AMDF). Consider, as an example, the following:

$$\frac{15}{32} \div \frac{3}{4} = \frac{15 \div 3}{32 \div 4} = \frac{5}{8}$$

This method can be explained by using a "missing divisor" strategy and connecting it to a conventional interpretation of fraction multiplication, from which a computation follows from the accepted rule:

$$\frac{15}{32} \div \frac{3}{4} = \frac{a}{b} \quad \Rightarrow \quad \frac{15}{32} = \frac{a}{b} \times \frac{3}{4} = \frac{a \times 3}{b \times 4}$$

$$\text{Thus, } a = 15 \div 3 = 5 \text{ and } b = 32 \div 4 = 8.$$

This method is simpler to apply when there is a divisibility relationship between the numerators and denominators, respectively. In such cases we refer to the pair of fractions as *compatible fractions* with respect to division. To elaborate, given $\frac{a}{b} \div \frac{c}{d}$, the AMDF is easily applied when $c|a$ (c divides a) and $d|b$. Our use of the term compatible fractions is in accord with the definition of compatible numbers, "which are number pairs that work easily together" (Van de Walle et al., 2016, p. 272).

The AMDF, while easily verified as correct computationally and algebraically, often leaves people with a sense of disbelief. It is not mentioned in North American (and, as far as we know, most European) curricula. However, Chinese teachers are not only familiar with the AMDF but also evaluate it as more efficient when the division involves compatible fractions. Ma (1999) considered teachers' familiarity with the AMDF, and their ability to apply it when reasonable, to be among the indicators of profound understanding of fundamental mathematics.

Since then, the AMDF has appeared in some studies of mathematics teacher knowledge. When presented to teachers in a study conducted by Tirosh (2000), the majority of participants initially considered the AMDF to be an error. Following instructional interventions that attended to participants' subject matter knowledge and pedagogical content knowledge of fraction division, participants still expressed a strong preference towards the "standard" algorithm.

Son and Crespo (2009) invited prospective teachers participating in their research to respond to a scenario related to teaching division of fractions. In this scenario, the AMDF was presented by a student, but then rejected by another student on the grounds that it is not the conventional method for dividing fractions. Son and Crespo investigated how "teachers reason and respond to a particular student's non-traditional strategy for division of fractions" (p. 243). Teachers' reasoning about the AMDF was analysed in terms of layers of reasoning (Campbell et al., 1998). Their responses to the student work were categorised as either teacher- or student-focused, with particular attention paid to the role of teachers' beliefs about mathematics teaching and learning as factors that guide teachers' response.

These studies highlight the importance of extending investigations into teachers' responses to students' non-traditional strategies. We continue this line of investigation by examining teachers' pedagogical reactions to the AMDF using Rowland et al.'s (2005) Knowledge Quartet framework.

The Knowledge Quartet

The Knowledge Quartet (KQ) is a practice-based theory of knowledge for teaching, introduced as a framework for the analysis and development of mathematics teaching (Rowland, 2020). The KQ (Rowland, 2020; Rowland et al., 2005), categorises the actions teachers use to bring their mathematical subject knowledge into play in the classroom. By developing this framework, Rowland et al. (2005) primarily aimed to provide teachers and their mentors with a means of retrospectively discussing their pedagogical choices related to teaching mathematics.

The KQ includes four elements: foundation, transformation, connection, and contingency. Foundational knowledge refers to a teacher's knowledge-in-possession, that is, the mathematical facts and beliefs that they have accrued through training and experience. Transformation and connection refer to a teacher's knowledge-in-action—the specific ways in which a teacher may leverage their foundational knowledge to prepare for and conduct teaching.

In particular, *transformation* concerns the ways teachers make their foundational knowledge accessible to students. This is accomplished through explaining concepts, demonstrating procedures, confronting, and resolving misconceptions, justifying, or refuting mathematical conjectures, and engaging students with exercise/practice. Transformation is exhibited in teachers' deliberate choice and use of examples, analogies, and/or representations.

Connection is revealed in teachers' practice when they aim for coherence within a lesson, or across a series of lessons. Connection is demonstrated through linking different meanings, descriptions, and representations of mathematical concepts, or procedures; anticipating the cognitive demand, complexity, or appropriateness of a concept; and ordering tasks and exercises to sequence instructions within and between lessons.

C*ontingency* differs from transformation and connection in that it refers to teachers' knowledge-in-interaction, which is revealed only in their response to an unexpected incident that concerns the teaching and learning of mathematics. Contingent incidents unsettle teachers and may prevent them from "making a surefooted and confident response at the time" (Rowland et al., 2015, p. 76). The main triggers of contingency include students' mathematical contributions, teachers' insight about their own practice, and the availability and functioning of educational tools.

Rowland et al. (2015) distinguished three different types of student contributions that might be unpredictable for teachers: a student's response to a question from the teacher, a student's spontaneous response to an activity, or discussion, and a student's incorrect answer. Teachers might respond to a student's unpredictable contribution in three different ways: by ignoring the student's idea, by acknowledging it but putting it aside, or by acknowledging it and incorporating it into the lesson. The three types of teacher responses are visualised in Figure 1.

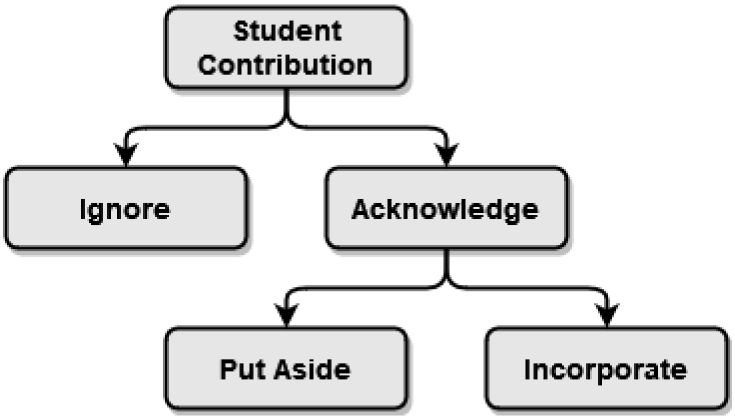

Figure 1. Rowland et al.'s (2015) categorisation of teacher responses to an unexpected student input.

As Rowland et al. (2015) noted, teachers do not often *ignore* student contributions except when "they seem not to have heard it, in which case the teacher's lack of response cannot be regarded as intentional" (p. 81). When a teacher *acknowledges* a student's input, she calls attention to the fact that the student has made an interesting, or original contribution. On the one hand, she might choose to *incorporate* this contribution by making it a meaningful part of the main mathematical activity of the lesson-in-progress, or by shaping the mathematical discussion so that other students also incorporate that idea into their mathematical knowledge. On the other hand, the teacher might *put aside* a student's idea, explicitly, or implicitly, if she does not leverage it in any way to augment the mathematics she had already planned for the lesson.

While the KQ framework was developed as a "theoretical tool for observing, analysing, and reflecting on actual mathematics teaching" (Rowland, 2020; p. 301), it is applicable for the analysis of teachers' responses related to how they imagine or describe an instructional situation. As such, in line with the stated goal of this article, using the KQ as a framework facilitated the examination of how teachers engage with a students' unexpected contribution—in this case, the use of the AMDF.

Research Question

The research addresses the following research question: How do teachers experience and respond to contingency when an alternative algorithm for fraction division is presented by a student? The research conducted was comprised of two studies. Study 1 involved prospective teachers ($n = 36$) writing about how they would respond to a student's suggestion to use an AMDF. In Study 2, teachers with various levels of professional experience ($n = 6$) were interviewed to capture their responses to a student who uses both an AMDF and a standard division algorithm. The teachers' responses from both studies were analysed by drawing on components of the Knowledge Quartet framework.

Study 1

Participants, Data Source and Analysis

Participants in Study 1 were thirty-six prospective elementary school teachers enrolled in an elementary mathematics methods course at a North American university. The goal of this course was to simultaneously grow prospective teachers' knowledge of elementary mathematics and corresponding pedagogical approaches by planning lessons and exploring resultant teaching situations. The data were comprised of participants' individual written responses to the *Fraction Division Task,* which included the prompt shown in Figure 2.

Imagine the following scenario:
You completed teaching a unit on multiplication and division of fractions.
You gave students several exercises to check their computational competence, including the following:

$$\frac{15}{32} \div \frac{3}{4}$$

Robin suggested the following solution:

$$\frac{15}{32} \div \frac{3}{4} = \frac{15 \div 3}{32 \div 4} = \frac{5}{8}$$

Figure 2. Fraction Division Task prompt.

As noted above, the AMDF is not typically included in North American mathematics curriculum. Thus, by presenting student work that uses the AMDF, the Fraction Division Task aims to engage teachers with an unfamiliar student response that deviates from the teacher's instruction. Therefore, in line with its description in the KQ, the appearance of this student-originated algorithm was treated as a contingent incident.

Our analysis began with one member of the research team coding the participants' submissions for elements of the KQ in line with their description by Rowland et al. (2015). For example, a participant who chose a specific example to demonstrate a property of the AMDF would be coded as demonstrating transformation; a participant who anticipates certain difficulties of the use of the AMDF would be coded as demonstrating connection. We note that these codes are not exclusive: a teacher who anticipates difficulties with a chosen example was coded as both transformation and connection. Finally, contingency was coded according to Rowland et al.'s (2015) three types of responses to unexpected student input (see Figure 1). A second member of the research team then coded each of the participant submissions with the same approach. Finally, the two researchers met to compare their codes and resolve any differences.

During this comparison process, it was determined that the three types of responses in Figure 1 did not completely capture the subtle differences between participants' responses to the AMDF. New codes were created to account for these differences (see Figure 3).

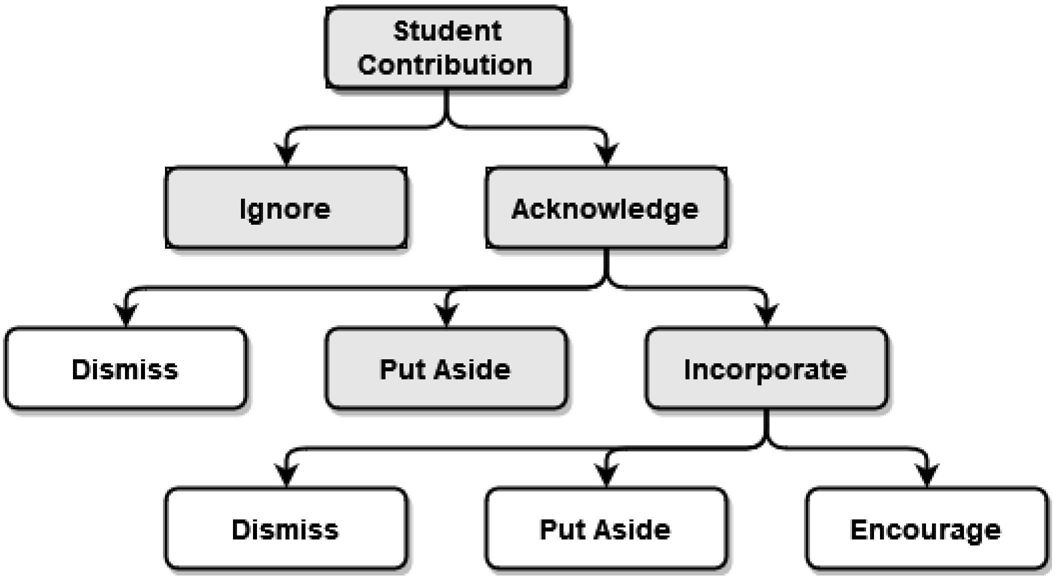

Figure 3. Additions (in white) to Rowland et al.'s (2015) responses to student input.

First, we observed that after acknowledging the student's use of the AMDF, some teachers immediately rejected it as mathematically incorrect. This is the *acknowledge and dismiss* category. Second, we observed that when teachers chose to incorporate the AMDF, they concluded their response with different teaching moves: either dismissing the unfamiliar algorithm, putting it aside, or encouraging its use in certain situations. These are the *incorporate and dismiss*, *incorporate and put aside,* and *incorporate and encourage* categories, respectively. The written responses were grouped into these expanded categories, which were compared and finalised until any disagreements were resolved.

Results

In the following subsections, the different types of teacher responses to contingency are introduced. Within each category, the frequency of its occurrence is provided, and exemplified by presenting relevant excerpts from the data set. All names that appear in excerpts are pseudonyms. Note, not all types of responses were evidenced. Response types that appeared in at least one written submission (see Table 1) are addressed. After introducing all the categories, we share further analysis of the teacher responses with respect to the transformation and the connection elements of the Knowledge Quartet.

Table 1
The Frequency of Response to a Student's Unexpected Contribution in Study 1

Type of response	Frequency
Ignore	0
Acknowledge	
Put aside	0
Dismiss	3
Incorporate	
Dismiss	17
Encourage	8
Put aside	8

Acknowledge and Dismiss

Three teachers recognised Robin's contribution but communicated to their students that the AMDF is an incorrect method and immediately introduced to them the standard algorithm as the proper way of dividing fractions. We categorised these types of responses as acknowledge and dismiss. Andrea's excerpt illustrates this category:

> I would tell Robin her method of calculating the numerator by the numerator and the denominator by the denominator would be correct if the question was asking her to multiply the fractions, as we have recently learned how to do …. I would then re-teach Robin individually, that when we divide fractions, we still need to multiply, but we also need to flip the numerator and denominator in the second fraction. I would ask her to try the problem again, this time changing the division symbol to a multiplication symbol and flipping the numerator and denominator in the second fraction.

Although Andrea acknowledged Robin's contribution, she interpreted the AMDF as an overgeneralisation of an algorithm for fraction multiplication to fraction division.

Incorporate and Dismiss

Seventeen teachers, who incorporated the AMDF in order to dismiss it eventually, instructed their students to always use one of the methods of fraction division that were introduced in class, such as the standard algorithm. These teachers justified this position by integrating the AMDF into the lesson's mathematical activity. The following excerpt from Daniel's response to the Fraction Division Task illustrates how he incorporated the AMDF in order to dismiss it:

> I will then give Robin an equation: $\frac{8}{9} \div \frac{3}{5}$ and ask her if she can use the same method for this equation. Robin will not be able to use the same method in solving this equation and therefore I will reiterate the importance of the multifactive inverse strategy …. I will specifically ask Robin to try and use a universal strategy, rather than doing extra work and figuring out if her own strategy works before using the strategy that was taught to the class.

As exemplified by Daniel's reaction, teachers whose responses fell into this category often highlighted that the AMDF is not ideal for problems with incompatible fractions. Although these teachers acknowledged that the AMDF would still result in a correct answer, some explicitly maintained that the method itself is still incorrect. Others did not treat the AMDF as incorrect, but as overcomplicated for most problems. In either case, they recommended their students to dismiss the AMDF.

Incorporate and Encourage

We categorised eight teachers' responses as incorporate and encourage. Similar to Daniel, these teachers posed another division problem with incompatible fractions and prompted Robin to solve it by using the AMDF. Unlike Daniel, however, the teachers in this category followed up this exercise by encouraging their students to use the AMDF whenever appropriate. Eva's response to the Fraction Division Task exemplifies this category. She explained how she would handle Robin's unexpected suggestion through an imaginary dialogue between two students and a teacher:

Ms. C: What if we tried another example? Can you explain to the class what you would do? (I wrote the following problem on the board: $\frac{7}{13} \div \frac{2}{3}$)

[Robin wrote the following:]

$$\frac{7 \div 2}{13 \div 3} = \frac{\frac{7}{2}}{\frac{13}{3}} = \frac{\frac{7}{2} \times \frac{3}{13}}{\frac{13}{3} \times \frac{3}{13}} = \frac{\frac{7}{2} \times \frac{3}{13}}{1} = \frac{7 \times 3}{2 \times 13} = \frac{21}{26}$$

…

Janice: Ms. C, can I show you how I solved this problem? I did it differently.

Janice wrote:

$$\frac{7}{13} \div \frac{2}{3} = \frac{7}{13} \times \frac{3}{2} = \frac{21}{26}$$

Janice: I just decided to invert and multiply right away. I took less steps than Robin, but we still ended up with the same answer.

After completing the dialogue, Eva provided a commentary which illustrates encouragement of the AMDF:

> In the end, I would encourage Robin to solve fractions in a way that makes sense to him, but I would also caution him that his approach is not always going to be so obvious. This would be when our dividends and divisors do not divide evenly into each other, such as our example problem of 7/13 ÷ 2/3. However, this is nothing to worry about, as we can still invert and multiply to get the answer.

In Eva's dialogue, a teacher-character gave Robin a problem featuring fractions that were not compatible; then, two student-characters solved this problem using two different methods but still arrived at the same answer. Incorporating both methods into the dialogue suggests that Eva would choose this new problem to show Robin that the AMDF would generate a correct answer despite the complications it poses in some cases. This approach was taken in most of the teachers' responses. A few teachers, however, used a problem with incompatible fractions to argue that the AMDF simply does not work in some cases. This led to two different ways in which teachers encouraged the use of the AMDF. On the one hand, some teachers left it to the student to choose which strategy to use for any given problem, as in Eva's commentary. On the other hand, some teachers explicitly stated that the AMDF should only be used for problems that involve compatible fractions, encouraging its flexible use.

Incorporate and Put Aside

We categorised eight teachers' responses as incorporate and put aside. As in the previous two categories, teachers first asked their students to use the AMDF to divide incompatible fractions. Unlike in these previous categories, however, the teachers did not then clearly conclude whether to dismiss the AMDF or encourage its use. For example, after demonstrating how to divide incompatible fractions by using the ADMF, Poh wrote the following:

> Now, I have gotten the chance to work this problem out and I noticed that it works but you are forced to use decimals within the fractions. This adds complication and is technically incorrect as decimals are a different representation of fractions.

Later in the same teacher's response, we see the following commentary:

> I would also make sure that all students who offered a solution, including Robin, felt validated for taking a risk and offering their solution to the class. This would hopefully reinforce the idea that, in math, there are many ways to the correct answer. It's not always about following a formula, or step by step solution, it's about problem solving and discovery.

As seen in the first excerpt, Poh stated that although the AMDF "works", it is also "technically incorrect" because it can lead to decimals within fractions. Despite this ambiguous evaluation, she never committed to completely dismissing or encouraging the use of the AMDF. In the second excerpt, Poh's strategy was instead to validate any attempts to solve the problem in the name of mathematical exploration and discovery. Although this is a commendable attitude, we interpreted Poh's position as a pedagogical shield (Kontorovich & Zazkis, 2016) in the sense that she is purposefully overgeneralising in order to avoid dealing with the specific question of the validity of the AMDF. By refusing to commit to a conclusion, Poh put aside the AMDF.

Teachers' Responses and the Knowledge Quartet

We identified the transformation element of the KQ in teachers' demonstrations of other division algorithms, such as when Andrea retaught the invert and multiplying strategy. Teachers introduced another procedure either when they thought that Robin needed to be reminded of, or taught the "correct" method, or when they wanted to compare a known strategy to the AMDF.

We also identified the transformation element of the KQ in teachers' deliberate choice of examples. All of the teachers who incorporated the AMDF in their response gave their students a problem which included incompatible fractions. Teachers chose this type of example to justify either that the AMDF always generates a correct answer, or that it "does not work" for some problems, as exemplified by Eva and Daniel, respectively. A majority of the teachers argued that the AMDF should not be used when the fractions in question are incompatible.

We identified various reasons which may have led to this claim. Some teachers stated that finding a correct answer by using the AMDF was a coincidence, while others interpreted the AMDF as an overgeneralisation of a fraction multiplication procedure to fraction division, as in the excerpt from Andrea's response. For some teachers, the method generates a solution in a non-standard form of a fraction—when the numerator, or the denominator is not a whole number—as in the excerpt from Poh's response. Some teachers evaluated these types of fractions as confusing for students, while others described them as incorrect answers. For some teachers, the AMDF did not even generate an answer because when the fractions are incompatible, the numerators, or the denominators "cannot be divided".

We identified the connection element of the KQ in teachers' anticipation of student difficulties while using the AMDF on incompatible fractions. This concern was revealed in teachers' responses from each of the categories involving incorporation. For some teachers, the difficulty was related to the fact that the AMDF required significantly more steps in these situations than the traditional algorithm, as in the excerpt from Eva's response. Others associated student difficulty with the appearance of decimals or fractions within fractions, as in the excerpt from Poh's response.

Even when teachers shared similar concerns about student difficulties or made the same judgement about the validity of the AMDF, their responses may not have fallen into the same category. For example, among the teachers who found that the AMDF "works" despite additional steps, some teachers (such as Eva) encouraged its flexible use. Other teachers dismissed the AMDF due to the complexity it added. Similarly, teachers who concluded that the AMDF "does not always work" responded to it differently. A majority dismissed the AMDF, while others encouraged its flexible use.

Summary of Study 1

In their written responses, eight prospective teachers incorporated the AMDF and encouraged its usage. However, the majority of participants responded by dismissing the AMDF, either without any mathematical justification or after incorporating the AMDF into their instruction to show that it is not applicable in all cases. Teachers who dismissed the AMDF reintroduced the standard algorithm in their response, either by teaching it from the beginning or by reminding the students of the main steps of the procedure. This suggests that these teachers assumed Robin did not know (or did not remember) the standard algorithm.

Additionally, given the extended amount of time that participants were given to complete the written task, it is unclear how participants experienced the actual moment of contingency, that is, what the participants' initial reaction was when the unconventional algorithm was proposed by a student. Only a few admitted that they initially evaluated Robin's calculation as wrong and were surprised to see the result was correct. We wondered whether such an experience was shared by other participants.

Towards Study 2

As indicated by our research question (Recall: "How do teachers experience and respond to contingency when an alternative algorithm for fraction division is presented by a student?), we were interested in teachers' experiences when the AMDF was unexpectedly suggested by a student, not only in their responses. So, in order to completely address our research focus, we designed Study 2 to investigate: (a) the spontaneous reaction of a teacher when the AMDF is first encountered; and (b) how this reaction changes when it is clarified that a student is also familiar with the conventional way of dividing fractions.

Study 2

Participants, Interview and Data Analysis

Six teachers with various levels of professional experience were recruited through convenience sampling to be interviewed. At the beginning of each interview, the teachers were invited to imagine themselves walking around a classroom to check students' computational competence after completing a unit on the multiplication and division of fractions. Then, teachers were shown the fictitious student work in Figure 4a and asked a series of protocol questions:

- What does this student work make you think?
- What would you do next if you encountered this student's work in the classroom?

Following the interviewees' responses to these questions, the interviewer asked follow-up questions that prompted the interviewee to expand on their answers. These secondary questions were necessarily dependant on the individual responses, but for example, if an interviewee noted that the hypothetical student skipped certain problems, he might then be asked to explain how he thinks the student chose which problems to skip. Note that, in this Figure 4a, the student appears to have chosen to solve three problems by using the AMDF. This aligns the first segment of the interview with the written Fraction Division Task in that it is not clear whether the student is familiar with and able to apply the standard algorithm.

Dividing Fractions	**Dividing Fractions**
1) $\frac{1}{2} \div \frac{8}{9} =$	1) $\frac{1}{2} \div \frac{8}{9} = \frac{1 \times 9}{2 \times 8} = \frac{9}{16}$
2) $\frac{15}{32} \div \frac{3}{4} = \frac{15 \div 3}{32 \div 4} = \frac{5}{8}$	2) $\frac{15}{32} \div \frac{3}{4} = \frac{15 \div 3}{32 \div 4} = \frac{5}{8}$
3) $\frac{4}{6} \div \frac{2}{9} =$	3) $\frac{4}{6} \div \frac{2}{9} =$
4) $\frac{3}{4} \div \frac{11}{12} =$	4) $\frac{3}{4} \div \frac{11}{12} =$
5) $\frac{8}{9} \div \frac{2}{3} = \frac{8 \div 2}{9 \div 3} = \frac{4}{3}$	5) $\frac{8}{9} \div \frac{2}{3} = \frac{8 \div 2}{9 \div 3} = \frac{4}{3}$
6) $\frac{1}{9} \div \frac{1}{3} = \frac{1 \div 1}{9 \div 3} = \frac{1}{3}$	6) $\frac{1}{9} \div \frac{1}{3} = \frac{1 \div 1}{9 \div 3} = \frac{1}{3}$
(a)	(b)

Figure 4. Snapshots from a fictitious student's notebook.

Next, the interviewer showed teachers Figure 4b and explained that the same student has now used the standard algorithm to solve the first question immediately after solving the three problems as seen in Figure 4a. The interviewees were then asked the second protocol question again, giving them the opportunity to revise their course of action based on the new information about the students' understanding. The interviewer used additional follow-up questions to elicit further elaboration from the interviewees when appropriate.

Each interview lasted for about 40 minutes. The interviews were recorded and transcribed, taking into account the participants' intonations, pauses, and exclamations. To code the interview transcripts, we identified contingency based on Rowland et al.'s description: contingent events are unsettling or disturbing for a teacher and prevent them from "making a surefooted and confident response at the time" (2015, p. 76). Therefore, teachers' statements that included words like "surprise" or that referred to their unfamiliarity with the AMDF were identified as evidence of a contingent situation. Teachers' evaluation of the algorithm first as wrong and then as right, or teachers' reflections on their own thinking about fraction divisions also helped us identify contingency. Teachers' responses to contingent incidents were coded by drawing on Rowland et al.'s (2005) transformation and connection elements of the KQ. We read the transcribed interviews several times, separately coded the contingent incidents and teachers' responses to them and compared codes until full agreement was reached.

In the following section, we report Valeria's and Mary's (pseudonyms) interviews as they illustrate the variety of teacher responses found within the total interview data. We chose these interviews in particular because they provide rich data that are also representative of the variety of reactions observed across the total body of interviews. Valeria had been teaching for at least five years at the time of the interview, Mary was a prospective teacher in the last term of her teacher education program.

Results

We present excerpts from Valeria's and Mary's interviews to illustrate the moment when contingency was triggered and the shift in the teachers' responses to the AMDF after they experienced a contingent incident.

Valeria

At the beginning of the interview, Valeria described the student work in Figure 4a by observing that "they just basically divide the top number by the top number and then bottom number by the bottom number." She inferred that this strategy was founded on a misconception related to the meaning of fraction. According to Valeria, the student perceived fractions as two separate whole numbers that did not have a relationship. She concluded that the student did not have a thorough conceptual understanding of fractions. In her response to this student, Valeria planned to reteach the meaning of fractions instead of introducing division.

> I feel that, in this case, I would feel it would be too soon for that student, developmentally, to learn about division of fractions. Let alone—I feel like they're not even ready to do operations with fractions of any kind. Because they lack conceptual understanding of what a fraction is. So, I would go to the very beginning, and I would try to distinguish between a whole number, and I would try to—just explaining to students what fraction is, is already quite difficult. And I would have to go back and say—and start with part to whole relationships. Just, I would start talking to them and maybe using some manipulatives to show that we always have to know what the whole is and then whatever is less would be a fraction of that whole. Which is a strange word for them because we don't encounter it in everyday life, except for maybe, "Oh, I bought it for a fraction of a cost." That could work.

In Valeria's initial response to the AMDF, we identified both connection and transformation dimensions of the KQ. The connection element is revealed when she addressed the student's developmental preparedness for doing operations with fractions. Accordingly, she referred to sequencing instructions when she said, "I would go to the very beginning … and start with part to whole relationships". This sequencing seems to result from Valeria's anticipation for the complexity of the fraction concept. We identified the transformation element of the KQ in Valeria's plan to use representations and analogies. She intended to use the former to explain the meaning of fraction, and the latter to link the unfamiliar mathematical term to a familiar situation from everyday life.

In this episode, we did not identify any uncertainty experienced by Valeria upon encountering the student's work. She seems to be surefooted in her interpretation of the student's work and her choice of appropriate teaching moves. Therefore, this initial response—disregarding the student's algorithm as a product of misconception—does not reveal a contingency.

After seeing the student complete the first problem (see Figure 4b), Valeria initially attributed the student's work to the memorisation of the standard algorithm and its application without understanding. Immediately after, her account indicates an insight about a different interpretation of the student work that she quickly disregarded:

> Valeria: Although now that I'm thinking. Maybe actually it is the other way around. Maybe [long pause]. Um. Nope. I think that's pretty much what it is. They've memorised the algorithm and they are good with multiplying whole numbers, so that's what they did.

After the interviewer prompted her to unpack her insight and solve the second division problem by using the standard algorithm, Valeria concluded that the AMDF provides a correct result:

> Interviewer: You also said that now I'm thinking that maybe this is the way around. What were you thinking when you said that? What made you think?
>
> Valeria: For a quick second I wanted to double check that, is it possible that this student already knows the invert and multiply algorithm and just is applying it skipping a step?
>
> Interviewer: Can you show it?

Valeria: That would be 60, and 32 times three … Oh, did I say 60? [mumbling, long pause] Oh! Oh, what's going on here? … [pause] I don't understand. This is clearly incorrect, why does it work? [pause]

Interviewer: Is there a relationship between this one [the AMDF] and this one [the standard algorithm] here?

Valeria: Well, judging by the results—yes! … if I invert and multiply here that would be 15 times four. So, I just un-invert? Is that's what's happening?

While examining the student's work, Valeria suspected that there is a relationship between the AMDF and the standard algorithm. She discovered that the AMDF, which she initially interpreted as a misconception, "actually works." This discovery puzzled her. The long pauses, the questions she asked, and her exclamatory reactions suggest that Valeria experienced an unexpected and unsettling incident. The presented algorithm challenged her knowledge about fraction division, triggering a situation of contingency. The trigger points to the teacher's awareness that something she thought was wrong is actually correct.

This new algorithm also challenged her previous response to the student work. Valeria continued by questioning her conventional approach.

> If I saw a kid trying to do this, I would stop them! I would say, "Wait, wait, wait, wait, wait. You don't want to do that. You're just dividing whole numbers. You need to divide the whole fraction." But then again, it works! So why would I want to stop them? Why would I want to teach them an extra step that is completely unnecessary, that utilises a different operation that I'm not even interested in teaching? I want to teach division, not multiplication right now. So why would I want to distract them from the division, into multiplication, when division works!? AUGH.

The questions in this excerpt indicate Valeria's uncertainty about how to respond to the AMDF. In her reflections, she first considered rejecting the student's response and teaching the standard algorithm. Immediately after, she problematised her traditional approach to fraction division while considering the coherence of her instruction. For Valeria, the focus should be only on division while teaching division. Therefore, introducing another operation, multiplication, would be not only unnecessary but also inappropriate as it may distract the student's attention from the focus of the instruction. Thus, her account reveals a recognition of conceptual appropriateness which is an element of connection in the KQ.

After questioning the need for the standard algorithm, Valeria solved the first problem by using the AMDF. In doing so, she confirmed that the AMDF produces correct solutions. She also noticed that, when the fractions are not compatible, applying the AMDF can be more difficult than the standard algorithm. The interviewer then asked Valeria what she would think if she saw a student using both methods, as in Figure 4b. Her response indicates her intention to incorporate the student's response into her lesson and to dismiss the standard algorithm:

> I want to throw away the invert and multiply algorithm. Why do we need it at all? … If there is a reason of the way and the algorithm is just to save time, at the expense of understanding, why do we need it? Instead of giving them a worksheet of 20 questions, just work with this one [the first problem]. … There's a whole lot of patterning happening here. I could probably spend an hour trying to figure out why this is so, and this is that and where the connections and where the relationships and it would be a worthwhile lesson just to do that. Look at the relationships, look at the connections. "Why is this, why is that and who needs the algorithm."

Valeria's response to the unexpected student work reveals two elements of the KQ: connection and transformation. We identified the element of connection in Valeria's attempt to prompt students to explore the relationship between the standard algorithm and the AMDF. We identified the element of transformation in Valeria's rethinking of her instruction by changing the amount, content, and purpose of the examples she would introduce to students. Instead of assigning numerous division questions to students through a worksheet as individual practice, she intended to incorporate one specific question into the classroom discussion to help students relate the two algorithms to each other. Through this connection, she justified a previous conjecture: the standard algorithm and the AMDF are the same, the former showing extra steps. At the end, she referred back to her question about the need for using the standard algorithm, indicating its dismissal once more.

Mary

When Mary saw Figure 4a, she assumed that the student skipped the questions because of having difficulty with them. As a response, she planned to ask, "What do you remember about dividing fractions? What's super special that we need to remember when we divide fractions? We need to flip the second number." Like Valeria, Mary does not yet appear unsettled by the AMDF. She seemed to attribute its use to the student's failure to remember the "correct" way of dividing fractions. Accordingly, Mary's questions address the invert and multiply procedure and seek to remind the student of its "special" rule. We associated this teaching move with the transformation element of the KQ, which includes demonstrating procedures.

When the interviewer asked Mary to describe the student work in Figure 4a, she stated that the student was "thinking about that top number and that bottom number as totally isolated numbers and not having a relationship. So not being one entity." In her response, she considered highlighting the meaning of fraction by adding brackets:

> I think from a logic standpoint, I get where they're coming from. So, like, "Oh, I'm dividing these numbers by each other." So, I think maybe putting a set of brackets around that number [the fraction 3/4] so that they're more easily able to identify that that's one whole thing. it's not 3 divided by 4. It's a whole thing, bigger, divided by another whole thing.

We identified the transformation element of the KQ in Mary's attempt to resolve a misconception by attending to the representation of the fraction. Mary identified a misalignment between the student's and her own interpretation of the fractional expressions. She inferred that the student perceived the fractional expression as the combination of two independent numbers instead of as a "whole thing" and aimed to resolve this misalignment by adding brackets.

When Mary was shown the student work on Figure 4b, her reaction indicated that it was an unexpected incident:

> Hmmmm! So, is this the same student who, they'd gone through. They had previously done 2, 5, and 6, and come back to it, and they've done number 1 but they'd done it with the correct [algorithm] …. Maybe that was feedback that I as the teacher had given. And they took that and then extended it into trying out that first one. Or maybe they sought help from a peer that was sitting next to them. There was some peer collaboration, and some solution happening there. So, I think the next step would then be to have a conversation about question number 1. What their strategy was, why they changed, what they were doing.

Mary seems to experience contingency upon seeing the student's solution to the first problem, probably because, within the same problem set, the student uses both a "correct" and an "incorrect" method. When Mary saw Figure 4b, she first reiterated the scenario to herself. This suggests that she was in disbelief that the same student was responsible for the use of both algorithms. Moreover, she speculated various reasons for the student's shift to the "correct" algorithm and planned to investigate the underlying reason for this shift. Mary's reaction in this excerpt suggests that the inconsistent student work was unexpected for her and that she did not have a surefooted explanation for it. Mary considered various possible responses to the student following this contingent incident:

> [I would] encourage them to try not only the ones that they haven't yet but go back and retry the original three questions that they had answered to see if they get a different answer. And then maybe do some checking in on which one do they think is more correct. … So, you could talk about the rationality of: "What does that fraction look like pictorially compared to the answer that you get." Yeah. And see if that's another way of explaining it to them that might help them come to a better solution. Like for example, Question 6, I think would be a really great example for that.

We identified the transformation element of the KQ in Mary's choice of different representations and examples to confront the student's use of the AMDF. She proposed two teaching moves which could prompt the student to reconsider using the AMDF: (a) to divide the fractions by using both algorithms and compare the solutions to find the "more correct" strategy; and (b) to check the reasonableness of the answer given by the AMDF by comparing it to the answer found by a pictorial representation of fraction division. She claimed that either of these teaching moves would be more effective with the sixth problem. Both moves also indicate an implicit conjecture made by Mary: "the student's alternative method is wrong, so it should generate incorrect answers." Accordingly, Mary's teaching moves seem to be aimed at dismissing the AMDF by incorporating it into her instruction.

When Mary applied the standard algorithm to the sixth problem, she realised that the AMDF gives the same result:

Mary: Three over nine, which is still one third. [pause] Yeah. Because one ninth multiplied by three over one would still equal one third. That's weird, because that one, the answer's the same no matter which way you do it!

Interviewer: You didn't expect that?

Mary: Nope! Oh nooooo! … I would have previously done the worksheet so I would have known to not use that one as the example for that statement.

[Mary solves the fifth problem by using the standard algorithm and concludes that it gives the same answer as the AMDF]

Mary: looking at the initial equation itself, my expectation was that when you're dividing two numbers less than one, the result should be lower. But it turned out that that's not entirely the case. Because one over nine [pause], three over one [pause], is three over nine, which is equal to one third. And then-- I'm getting the same answer as the student, but I don't know how I'm doing that.

Interviewer: So do you think you make a mistake in your calculations?

Mary: I think so, I just am not 100 percent sure where it is.

[Mary again solves question five with standard algorithm and concludes that the answers are the same]

Interviewer: Hmm. It sounds like you believe that they shouldn't be equal. What makes you think that way? Why do you think this algorithm shouldn't give the same answer as YOUR algorithm?

Mary: Because I was taught in Grade 7 in order to divide a fraction, you have to flip and multiply it. So, my prior knowledge is telling me it shouldn't work. Because there's only one way to accurately write and work through these fractions. Fifteen over 22 multiplied by four over three [pause] and it's 15 over [mumbling, "solving problem"—reading off her work in numbers]. Which is five over eight. They got the questions right! That's so funny. Then why do we tell them to kiss and flip it [invert and multiply]? That's so much more work!

Finding that the two algorithms provide the same answer appears to be yet another experience of contingency for Mary. Her description of this finding as weird and her initial exclamations indicate that she was still unconvinced that both algorithms give the same answer. After solving a second problem with the standard algorithm, she re-evaluated her conjecture that the quotient should be smaller than the dividend. Even so, Mary attributed her finding to a computational error and solved the fifth and the sixth problems by using the standard algorithm. After repeatedly double checking her answers, she stated with more confidence that the student's algorithm, which she initially evaluated as incorrect, gives the correct solution. Like in Valeria's case, the trigger of contingency seems to be the teacher's awareness that "something is correct."

After realising that the AMDF generates correct answers, Mary wondered why she should teach the standard algorithm. She found an answer to this question when she concluded that the AMDF is more difficult when the fractions are not compatible:

> Wait, did I figure out why they were uncomfortable dividing this next thing? I feel like some of the other ones, when you divide them, they're not as seamless. Like, six divided by nine is [pause] weird. And four divided by 12, or three divided by 11, that's why they didn't do those questions. But now that they have-- and that's why you would show both options, or why you would want both options to be present. It's so that, if the number on one side is larger, you would want the option to then flip

> and multiply out rather than have a fraction within a fraction that you have to then simplify for. That's why. Okay. I now understand why this student avoided those questions at first.

We identified the connection element of the KQ in Mary's anticipation that the AMDF may lead to a complicated expression which requires additional computations. Considering this complexity, she planned to incorporate the AMDF into her instruction as one of two division methods which should be selected flexibly based on the compatibility of the fractions involved in the problem.

Summary of Study 2

Our interviewees initially dismissed the AMDF as wrong and considered either re-teaching the fraction concept or introducing the standard method for fraction division. Once they saw that the same student used both the standard algorithm and the AMDF and realised that the AMDF actually provides a correct result, the interviewees revised their initial response. Valeria incorporated the AMDF and considered dismissing the standard algorithm, while Mary considered how the AMDF can be incorporated into her instruction to encourage its flexible use.

Discussion and Conclusion

In our work, we drew on the theoretical notions from Rowland et al.'s (2005, 2015) Knowledge Quartet to examine teachers' engagement with a student's use of an unconventional algorithm for fraction division in two different contexts: through their written reports and interviews. The following research question guided our investigation:

> *How do teachers experience and respond to contingency when an alternative algorithm for fraction division is presented by a student?*

While only a few of the written reports explicitly acknowledged an unsettling incident experienced by the participant, the interviewees were all surprised that the AMDF gives a correct answer. Their surprise was not immediate; however, it emerged once the interviewees gained more information about the student's knowledge.

Our work illustrates a difference between teachers' planned and spontaneous responses to contingency as captured via written submissions and interviews, respectively. While several teachers incorporated the AMDF in order to dismiss it or put it aside from the beginning of their written reports, the interviewees revised their response to the student work after experiencing contingency. Initially, they acknowledged and dismissed the AMDF; later, they incorporated and encouraged it. Our results show that even though teachers could correctly evaluate student-generated strategies when they have more information about students' thinking, during instruction, they may miss some opportunities to capitalise on students' mathematically valid responses with which they are not familiar. This finding is similar to those of Magiera et al. (2013), in which teachers' engagement with student work also differed between written and in-person contexts.

Both written submissions and the interviewees' responses revealed the transformation and connection elements of the KQ. We identified the transformation element of KQ in teachers' demonstration of the standard algorithm and in their choice of different representations and examples to confront the student's use of the AMDF. The connection element of the KQ appeared when teachers highlighted the relationship between the AMDF and the standard algorithm. However, these teachers constituted the minority of the participants, which echoes the findings in Son and Crespo (2009). The connection element of the KQ was also revealed in the teachers' concern that applying the AMDF with incompatible fractions may pose some difficulties for students. Teachers addressed this concern in different ways across their responses. Some dismissed the AMDF completely, whereas others encouraged its flexible use according to the fractions involved in the problem.

This finding shows that while engaging with student contributions, teachers generally integrate similar teaching moves into their responses. However, despite drawing on the same elements of the KQ, responses to contingency vary among teachers. Moreover, as captured during the interviews, the same teacher might adapt their response to the same unconventional student contribution when their assessment of the students' mathematical proficiency changes, in line with the findings of in Wager (2014). Overall, we identify a deficit approach in teachers' evaluation of unconventional student strategies. This is similar to the findings in Hines and McMahon (2005). Once the teachers can shift this perspective, their evaluation of both the algorithm and response to the student might shift as well. So, as an educational implication, this study highlights the importance of helping teachers avoid rushed judgement towards students' unconventional strategies, in addition to expanding teachers' mathematical and pedagogical knowledge. We consider this as a contribution to teachers' professional knowledge.

In addition, this study contributes to research methodology and theory. In terms of methodology, we expanded the applicability of the KQ. Rowland et al. (2005, 2015) used this framework to analyse classroom events that unfold based on teachers' prior planning. We used it in the analysis of a teacher's potential teaching approaches as described in their written work and as they spontaneously emerged in an interview. In terms of theory, we exemplified a nuance of Rowland et al.'s (2015) categorisation of triggers of contingency and provided a further expansion related to teachers' responses to unexpected incidents. The two contributions to the theoretical framework address the remark made in Rowland et al. (2015) that the provided categorisation is not complete and thus has potential to be expanded.

Our findings show that teachers' awareness that a student's work was actually correct unsettled them and challenged their previous response to the AMDF, triggering contingency. This type of trigger is different from Rowland et al. (2015), where the contingency resulted from teachers' awareness that a student's work was actually wrong. In both cases, the teachers identified a mistake in their teaching, which is one of the categories identified in Rowland et al. (2015). We argue, however, that it is important to distinguish the types of mistakes that trigger contingency in practical settings. In Rowland et al. (2015), the teacher's mistake was agreeing with a mathematically wrong statement. In our findings, the initial mistake was disagreeing with a mathematically correct statement. Further, if the interviewees had not seen the student work in Figure 4b, they may not have had such an insight. Therefore, teachers' unfamiliarity with the AMDF, the extent of information they can access about student thinking, and new teacher insights all played a role in triggering contingency. We suggest that exposing prospective teachers in their teacher preparation courses to student work that is correct but unexpected would help them moderate their reactions in the future and lead them to examine, rather than dismiss, student responses that deviate from their instruction.

Our findings also captured nuances in teachers' responses to unexpected student contributions that were not highlighted by Rowland's original categories: ignore, acknowledge and put aside, and acknowledge and incorporate. We expanded Rowland et al.'s (2015) categorisation by adding a new category—acknowledge and dismiss—and elaborating on the acknowledge and incorporate category by examining whether the teacher accepts the student contribution after incorporating it. We observed that teachers might incorporate and put aside, incorporate and dismiss, or incorporate and encourage the student's idea. This expanded categorisation can provide teacher educators with a richer framing to help teachers analyse, reflect on, and improve their responses to student contributions.

We observed a variety of reactions to the AMDF that sheds new light on prior research. In Study 1, some of our participants' responses are in discord with Tirosh (2000), which showed prospective teachers' tendency to prefer the standard algorithm over the AMDF. This difference may be explained by the fact that teachers' professional obligations can influence their instructional decisions (Herbst et al., 2016). In Study 2, Valeria and Mary had different levels of professional teaching experience and also demonstrated different responses to the AMDF. Based on these observations, future research should focus on the role of teachers' professional obligations and teaching experience in their responses to unexpected student work. The categorisation of teachers' responses to student-triggered contingent events has potential to expand further according to the teachers' professional positions and training.

We conclude by presenting the limitations of each study. In Study 1, the participants responded in writing to the alternative algorithm, which allowed them time to explore this algorithm and carefully plan their response. On the one hand, the majority of the written responses do not provide sufficient insight about how the teachers experienced contingency and how they would have reacted if they encountered the AMDF in their actual classrooms. On the other hand, the interview participants of Study 2 responded immediately upon seeing the AMDF without having time to familiarise themselves with the particular division problems, which teachers generally do while planning their lessons. Even though interviewing teachers helped us witness how teachers experienced contingency during a friendly conversation, it may not have provided totally reliable information on teachers' responses in an actual instructional interaction. This is because in an actual teaching situation they could have been better prepared and more conscientious about how to respond to a student.

Nevertheless, approximating teaching practice in any context may help teachers become aware of their tendencies while responding to student-invented strategies. By engaging with the AMDF, participants realised their implicit assumption that the standard algorithm is the only way to divide fractions. Realising this assumption, some participants commented that engaging with the AMDF became a learning opportunity for them, which would prompt them to think critically while responding to students' alternative strategies in the future.

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Competing interests

The authors declare there are no competing interests.

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