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Design Principles for Teacher Investigations of Student Work

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This article describes a specific approach to a teacher investigation of students' written work. The aim of the investigation was for in-service middle school teachers to externalize their thinking about their students' mathematical thinking and to then reflect upon these externalizations and make refinements accordingly. This aim mirrors the purposes of student-level mathematical activities entitled model-eliciting activities. Therefore, the teacher investigation was patterned after the design principles for student-level model-eliciting activities. These design principles and how the teacher investigation was modeled after these principles are described. In addition, the teachers' externalizations, reflections upon, and revisions to their thinking about their students' thinking are explained. The article concludes with general design principles that other teacher educators can use in designing their respective professional development contexts.

Many of the latest reform efforts in mathematics education stress the importance of teachers attending to and understanding their students' mathematical thinking (Ball, 1997; Maher & Martino, 1992; NCTM, 2000; 2001). However, Ball (1997; 2001) and Schifter (2001) explain that focusing on students' thinking in the classroom while maintaining mathematical integrity can prove difficult. In addressing such difficulties, Ball (1997) explains that teachers' investigations of artifacts of teaching and learning, such as students' written work, hold "promise for equipping teachers with the intellectual resources likely to be helpful in navigating the uncertainties of interpreting student thinking" (p. 808).

In this article, a specific approach to an investigation by teachers of their students' written work is described. The teacher investigation was conducted as part of the on-going Professional Development School collaboration between a Midwestern university and a local middle school and was facilitated by the author[1]. The aim of the investigation was for in-service middle school teachers to *externalise* their thinking about their students' mathematical thinking and to then *reflect* upon these externalisations and make *refinements* accordingly. This aim mirrors the purposes of student-level mathematical activities entitled *model-eliciting activities* (MEAs) (Lesh, Hoover, Hole, Kelly, & Post, 2000). Therefore, the teacher investigation was patterned after the design principles for student-level MEAs. This article will begin by explaining the MEA design principles and will then explain how the teacher investigation was modeled after these principles. It will

[1] The facilitation coincided with the author's dissertation study which examined the teachers' collective interpretations of their students' thinking as a result of participating in these investigations of students' work.

also describe the teachers' externalisations, reflections upon, and revisions to their thinking about their students' mathematical thinking.

Design Principles for Student-Level Model-Eliciting Activities

MEAs are designed to help middle school students develop conceptual foundations for deeper and higher order ideas in pre-college mathematics (Lesh, et al., 2000). Each activity asks students to mathematically interpret a complex real-world situation and requires the formation of a mathematical description, procedure, or method (i.e., a model) for the purpose of making a decision for a realistic client. Because students are producing a model (instead of just a one-word or one-number answer) and working in groups, they externalize and reveal their thinking throughout the activity and in their final solutions. Such externalisation of their thinking assists students in reflecting upon how well their current thinking strategies solve the problem and in making appropriate revisions to their solutions. MEAs are similar to other mathematical modeling projects that have taken place in several parts of the world including, among others, Australia, Denmark, Germany, Netherlands, and the United Kingdom (Blum & Niss, 1991).

The *Departing On-Time MEA*[2] presents students with departure times for five different airlines (see Appendix A). The client, the Ridgewood High School Spanish Club, needs help selecting an on-time airline for an upcoming study abroad trip. Thus, students have to develop a procedure for ranking the five airlines from most likely to least likely to depart on time. The activity is specifically designed to help students develop conceptual foundations for statistical concepts such as mean, standard deviation, spread, and frequency.

To develop MEAs, designers rely on six design principles (Lesh et al., 2000). The first principle is called the Model Construction Principle, which ensures that the activity requires the construction of an explicit description, explanation, or procedure for a mathematically significant situation. Such products externalise how the students interpret the situation and also reveal the types of mathematical quantities, relationships, operations, and patterns that students take into account. In the Departing On-Time MEA, students are specifically asked to "Develop a *procedure* for ranking the five airlines from most to least likely to depart on-time."

The second design principle is the Reality Principle (or the *personal meaningfulness* principle). This principle ensures that students can interpret the activity meaningfully from different levels of mathematical ability and general knowledge. In the Departing On-Time MEA, students relate to the idea that they want to select an airline that is more likely to depart on time, and they quickly realise that lower numbers in the table indicate more punctuality.

The third design principle is the Self-Assessment Principle, which ensures that the activity contains criteria the students themselves can identify and use to test and revise their current ways of thinking. While working on the Departing On-Time MEA, students can return to the data to self-assess whether the results of

[2] It is recommended that readers work through the problem before continuing to read.

their calculations seem to reflect what they visually see in the data. For example, when students find that the average number of minutes late for the airlines are all nearly the same, they return to the data and notice that the airlines do differ in their frequency of being late. Such reflections typically lead the students to revise their thinking and to count the number of on-time flights for each airline.

The fourth principle, the Model Documentation Principle, ensures that students are required to create some form of documentation that will reveal explicitly how they are thinking about the problem situation. Such documentation is beneficial for the teacher because it reveals how the students are interpreting the given situation. It is also beneficial for the students because it becomes easier for the students to visualize and thereby reflect on their thinking. In the Departing On-Time MEA, the problem requires students to produce explanations, procedures, or descriptions as part of their solution and to explain their solutions in written letters.

The fifth principle is the Construct Share-Ability and Re-Usability Principle, which requires students to produce share-able and re-usable solutions. By asking the students to produce products that can be used by others beyond the immediate situation, students go beyond personal ways of thinking to developing more general ways of thinking, often resulting in more powerful mathematics. In the Departing On-Time MEA, the students are to develop a ranking procedure to be shared with the Spanish Club and to make their procedure general so the Spanish Club may use the procedure to rank additional airlines."

The sixth principle, the Effective Prototype Principle, ensures that the model students develop will be as simple as possible yet still mathematically significant. The goal is for students to develop solutions that will provide useful prototypes for interpreting other similar situations. In the Departing On-Time MEA, the problem scenario is clear and simple – find the airlines that are more likely to depart on time, and the students' solutions usually provide a useful prototype for interpreting other situations in which frequency counts are appropriate.

The Teacher Investigation

The purpose of the teacher investigation modeled after these six design principles was for the teachers to express, reflect upon, and make revisions to their thinking about their students' mathematical thinking. Due to the thought-revealing nature of MEAs, the teachers examined samples of their students' work from the Departing On-Time MEA.[3] The investigation was centered around the development of a *Students' Thinking Sheet* (STS), which outlines the different mathematical approaches used by students in solving the MEA, includes excerpts of students' actual work, outlines the mathematics involved in each of the students' approaches, and describes the effectiveness of each approach (see Appendix B).

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[3] Other thought-revealing mathematical activities that elicit multiple solution strategies would also suffice for similar teacher investigations.

The teacher investigation began with an introductory workshop, where the teachers worked together to complete the Departing On-Time MEA. Following the workshop, each teacher implemented the MEA in his or her classroom. Then, each teacher created an individual STS by synthesizing the information in his or her observation notes and in the students' work. Finally, at a second teacher workshop, the teachers shared their individual STSs and then worked together to create one consensus STS. As they created the consensus sheet, they kept their purpose for creating the sheet in mind - to explain the students' ways of thinking so that their colleagues could later interpret in advance how students typically solve the MEA. The Departing On-Time teacher investigation was the second in a series of five similar investigations centered around MEAs and was captured through the use of video-recording and the collection of teacher products created during the investigation.

Design Principles for the Teacher Investigation

The purpose of the teacher investigation was for the teachers to express, reflect upon, and revise their thinking about their students' mathematical thinking. Thus, in designing the investigation, the design principles for student-level MEAs, which require students to express, test, and revise their mathematical thinking, were mirrored. Also taken into account were teacher-activity suggestions provided by Doerr and Lesh (2003).

For the Model Construction Principle, the teachers should be asked to develop a description, explanation, procedure, or justified prediction for interpreting their students' work; therefore, the teachers were asked to prepare the STS. Creating this sheet required the teachers to formulate an explanation of their students' mathematical thinking and served as a prediction of how future students may solve the associated MEA.

For the Reality Principle, the teacher investigation had to a) allow the teachers to begin the activity using their existing knowledge and b) require the teachers to investigate a realistic situation, i.e. to interpret a situation within the context of their own practice. For this investigation, the teachers examined their own students' work for the purpose of creating the STS. The teachers were readily prepared with their existing knowledge of their students and of mathematics to immediately begin interpreting their students' work, and the investigation was clearly situated within the context of their practice.

For the Self-Assessment Principle, the teacher investigation had to ensure that the teachers themselves could judge if their interpretations, ideas, or models (such as a STS) were meeting their desired purpose. In this teacher investigation, the teachers self-assessed their interpretations of their students' thinking by returning to the examples of their students' work. Also, they assessed their STS for its intended purpose – to capture and explain the students' ways of thinking such that their colleagues could later interpret in advance how students typically solve the MEA.

To meet the fourth design principle, the Documentation Principle, the activity had to require the teachers to produce documentation that would reveal the teachers' thinking throughout the activity. As a result of engaging in the investigation, the teachers created documentation of their thinking about their students' thinking as they developed their individual and consensus STSs. Such documentation was both in the sheets themselves and in the process of sheet design.

To meet the fifth design principle, the Share-Ability and Re-Usability Principle, the investigation had to require the teachers to develop products (such as the STS) that were shareable with other teachers or other interested parties and that were reusable for themselves, for other teachers, or for other interested parties. The teachers met this principle by purposely designing the consensus STS to be shared and re-used by their colleagues. Specifically, the teachers believed that their colleagues would re-use the STS next year not only to learn in advance how students solve the Departing On-Time MEA, but also to evaluate their students' solutions and to share with parents at conferences. So, the clients (to parallel student-level MEAs) were other middle school mathematics teachers.

To meet the sixth design principle, the Effective Prototype Principle, the activity had to require the teachers to develop a solution that would serve as a useful prototype for interpreting other samples of students' work. The teachers found that the process of producing the STS (i.e., organizing and classifying the students' thinking according to different solution strategies) served as a prototype for organizing their interpretations of their students' thinking on successive MEAs. By the Departing On-time teacher investigation, the prototype had gelled into a typical sequence. First, the teachers took turns sharing the thinking of their students on the associated MEA. As the teachers shared, one of the teachers served as a recorder and placed the descriptions of the students' thinking in rough categories. When the teachers exhausted the different types of students' thinking, they returned to the rough categories and made refinements by comparing and contrasting the various solution strategies. It was at this point that the teachers engaged in the majority of reflecting upon and revising their interpretations of their students' thinking. Ultimately, the teachers arrived at three or four main categories describing the students' thinking on the respective MEA. With these categories delineated, the teachers moved on to discuss the mathematics and the effectiveness of each solution strategy to finish the development of their Consensus STS.

Expressing, Reflecting, and Revising

Modeling the teacher investigation after the student-level design principles did lead the teachers to progress through a collaborative process of expressing, reflecting upon, and accordingly revising their thinking. Specifically, during the Departing On-Time investigation, the teachers began by sharing (expressing) their interpretations of how their students thought about and solved the Departing On-Time MEA for the purpose of creating the STS as per the Model Construction Principle. As the teachers shared their interpretations, one of the teachers recorded the students' thinking strategies on a poster sheet to begin meeting the

Documentation Principle. Eventually, the teachers had compiled a list of seven different strategies:

1. Find the total number of minutes late for each airline and rank the airlines from lowest total to highest total.
2. Find the percentage of on-time flights for each airline and rank the airlines from highest percentage to lowest percentage.
3. Find the average time late for each airline and rank the airlines from lowest average to highest average.
4. Find the average time late when flights were late for each airline and rank the airlines from lowest average to highest average.
5. Redefine late as consisting of flights that leave more than 5 minutes late.
6. Count the number of on-time take-offs for each airline and rank the airlines from the highest total to the lowest total.
7. Redefine late as consisting of flights that leave 30 or more minutes late.

When the teachers felt they had exhausted the different ways of thinking about the Departing On-Time MEA, they were prompted to consider their purpose for listing these ways of thinking – to create a consensus STS to be shared with their colleagues, a purpose prompted by the Share-Ability and Re-usability Principle. This consideration encouraged the teachers to begin looking for an easier and more precise way to describe the different ways that the students thought about the activity. Specifically, the teachers began to decrease their list to three or four main ways of thinking. They looked for commonalities and differences between the seven strategies (a process of reflecting upon their externalised thinking made possible by the Self-Assessment Principle) and subsequently revised their thinking.

First, the teachers quickly classified strategies 5 and 7 as corresponding to a main way of thinking, that of redefining late to be some time period greater than zero but less than thirty minutes. Also, the teachers eventually decided that students would be likely to calculate the average-number-of-minutes-late-when-flights-are-late only after going through the conceptual process of redefining late. Therefore, the teachers decided that strategy 4 was a subset of strategy 5 and 7.

Second, the teachers examined strategies 2 and 6. They realized that finding the percentage on-time for each airline was only two computation steps (dividing by 30 and multiplying by 100) beyond counting the number of on-time flights for each airline. Thus, they determined that the two strategies relied on the same underlying concept, that is, an airline with a lower incidence of late flights would improve the Spanish Club's chances of catching their connecting flight. The teachers decided to therefore combine the two strategies.

Finally, the teachers compared strategies 1 and 3. They again determined that the two strategies relied upon the same idea – airlines with fewer minutes late improve the chance of making a connecting flight. The teachers realized that finding the average number of minutes late per flight was only one calculation (dividing by 30) beyond totaling the number of minutes late. They also recognized that the rankings from the total number of minutes late and from the average number of minutes late were the same because all of the airlines contained 30 flights.

After revising their interpretations of their students' thinking, the teachers arrived at the three main ways of thinking described in their STS: a) finding the total or average number of minutes late, b) finding the ratio or percentage of on-time flights, and c) redefining late. The teachers arrived at these revisions by considering the underlying conceptual idea for each strategy rather than focusing on the types of calculations made by the students. Figure 1 contains the teachers' poster sheet after they had consolidated similar strategies (see the parenthetical remarks at the end of each line describing how the 7 initial strategies were combined into the three main ways of thinking). This poster sheet served as the model for the STS mentioned previously.

Strategies

1. Total minutes late (see 3)
2. % "on-time" (see 6)
3. Average time late (x ÷ 30)
4. Average time late on only late arrivals (see 7)
5. > 5 minutes late (average, sum) (see 7)
6. How many "on-time" takeoffs
7. Define late as greater than or equal to 30 minutes

Figure 1. Teachers' poster sheet describing the main ways of thinking exhibited by their students on the Departing On-Time MEA.

Conclusion

The STS investigation led the teachers to *express*, *reflect* upon, and *revise* their thinking because it was designed by modeling the student-level MEA design principles. The Model Construction and Documentation Principles ensured that the teachers externalised and documented their interpretations of their students' thinking, thereby making reflection upon and revision to their interpretations more likely. The Reality principle, by engaging the teachers in an investigation embedded in their own practice, enhanced the teachers' dedication to achieving their aim of producing a STS. The Self-Assessment and Share-ability and Re-usability Principles prompted the teachers to revise their interpretations of their students' thinking. In particular, the teachers looked beyond computation and calculation differences in their students' thinking to take into account the underlying concepts. The ability to identify the concepts with which students are working is crucial as it assists teachers with making instructional decisions based on their students' current understandings. Finally, as a result of the Effective Prototype Principle, the teachers developed a prototype for organizing, classifying, and making sense of their students' thinking on thought-revealing mathematical activities. Such a prototype assists teachers with interpreting their students' thinking on various mathematical activities.

The teachers also learned that students can approach tasks such as MEAs using different ways of thinking and that not all mathematical tasks have just one "right answer". As the teachers reported in a handbook about MEAs, "

> MEAs do not typically have one right answer; however, some solutions are better than others. Thus, when assessing your students' work or providing feedback to your students, it is more helpful to ask, 'How well does this particular solution meet the needs of the client?' instead of asking, 'Is this the right answer?'

Patterning the teacher design principles after the student-level MEA design principles proved fruitful for this investigation, both for encouraging teacher development and providing documentation about teacher development. As such, other teacher educators will likely find these principles useful in their professional development contexts. In the meantime, research is necessary to understand more fully just how engaging teachers in such investigations may impact their abilities to make sense of their students' thinking within the classroom and whether such investigations help teachers deal with the complexities identified by Ball (1997; 2001) and Schifter (2001). As we await such research, Table 1 provides questions for each design principle that teacher educators can consider in planning their respective teacher investigations and professional development.

Table 1
Verification Questions for the Six Design Principles for Teacher Investigations

Construction Principle	Are teachers required to make constructions, explanations, assessments, and/or decisions about the students' work and thinking?
Reality Principle	Are teachers asked to make sense of the students' work and thinking based on extensions of their own personal knowledge and experiences? Does the task require teachers to interpret situations within the context of their actual practice?
Self-Assessment Principle	Is the purpose clear so teachers themselves can judge if their interpretations are bringing about desired consequences in particular contexts? (Doerr & Lesh, 2002) Will the teachers be able to judge for themselves when they need to revise or extend their interpretations of the students' work and thinking?
Documentation Principle	Will completing the task require teachers to document how they are thinking about the students' work and thinking?
Share-Ability and Re-Usability Principle	Are teachers required to develop products that can be shared with other teachers and researchers and that can be re-used by themselves, other teachers, or other interested parties?
Effective Prototype Principle	Does the solution provide a useful prototype for interpreting other students' work and thinking?

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Credits

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Appendix A: Departing On-Time Model-Eliciting Activity

In June, Ridgewood High School's Spanish club is going on a study abroad trip to Venezuela. Last year, when traveling to Barcelona, their connecting flight to Reykjavik, Iceland was late. They missed their connecting flight and had to stay overnight in the airport! This year the club is being more careful. So far, they have identified five airlines that fly from O'Hare Airport to Venezuela, but they are still identifying more airlines. The flights all have a connecting flight in Mexico City. The students want the airline with the smallest chance of departing late from O'Hare so they can catch their connecting flight. Below is information about departure times for flights leaving O'Hare and arriving in Mexico City for each of the five airlines. Develop a procedure for ranking the five airlines from most to least likely to depart on-time. Describe your procedure in a letter to the Spanish Club so they may use your procedure with additional airlines.

Sky Voyage Airline	Central American Airline	Mexico Express	Sudamerica Internacional	Southeast Airline
5	15	9	0	0
0	9	5	25	5
20	4	5	0	0
5	0	5	9	9
0	0	125	0	40
6	14	10	0	0
0	20	5	4	5
0	15	10	0	25
15	16	0	35	10
0	0	4	0	30
0	0	10	0	12
7	15	10	10	0
0	10	10	5	0
5	10	9	55	10
40	25	7	0	9
4	5	12	0	5
0	20	5	0	0
0	15	0	17	27
0	11	10	5	11
0	12	7	0	0
3	0	13	65	30
60	5	0	5	5
5	0	0	0	0
0	30	10	0	4
7	4	5	2	40
0	5	4	0	0
0	10	6	0	15
123	10	5	75	0
0	25	7	0	6

5	4	5	0	9

Appendix B: Students' Thinking Sheet for the Departing On-Time Model-Eliciting Activity

Description of Solution Strategies	Mathematics	Effectiveness
Strategy #1: Students may find the total number of minutes late for each airline and then find the average number of minutes late for each airline. Finally, they rank the airlines from lowest to highest We have been looking over various Airlines' flight times from the month of June in 1999. We added up the total amount of minutes they were late and divided it by 30 to find the average amount of minutes a particular airline was late per day. *Average Number of Minutes Late:*	Adding the minutes; finding the average number of minutes; comparing and ordering the averages or total minutes to rank the airlines; and rounding the averages to the nearest tenth.	Easy to implement, but not very effective. A few very late departures by one airline will be equal to several minimally late departures by another airline. Also, the averages only differ by a few seconds.
Strategy #2: Students may count the number of times that each airline is on-time (count the zeros). Then, they may find either the ratio of on-time flights or the percentage of on-time flights for each airline. Finally, they rank the airlines from highest to lowest. We found our answer by counting how many times the airlines were on time. We picked the one that was on time the most. *Counting Number of On-Time Flights:*	Adding the number of on-time flights; finding the ratio of on-time to total departures; finding the on-time percentage; comparing and ordering to rank the airlines.	Easy to implement and more effective than strategy #1, but it does not take into account that departing up to 20 minutes late may still be okay for catching a connecting flight.

Strategy #3: Students may redefine late as occurring when a flight departs more than 5-20 minutes late and incorporate this new definition into strategy #1 or strategy #2. For example, they may find the average number of minutes flights are late when they are late, or they may count the number of times that each airline departs within 5-20 minutes or less. *Counting Number of Flights Departing within* If you are wanting to know the probability of the airlines being late, you must find the ratio. We did this by not counting 5 minutes and under as being late. Sky Voyage Airline's ratio of being late, for instance, is 22/30, 22 being the times the airlines is not late and 30 being the amount of days. *Fewer than 5 Minutes:*	Determining an acceptable range for departure times; same mathematics as strategy #1 or #2.	A little more complex to implement and more effective than strategy #1 or #2 because it allows for minimally late departures. Its effectiveness depends on identifying an acceptable range for departure.