Lesson Plays as a Mirror on Prospective Teachers' Professional Knowledge for Mathematics Teaching

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In this study, we analyse 23 prospective teachers' lesson plans for short whole-class mathematics discussions held with students at a middle school. The plans are in the form of "lesson plays", where an imagined discussion between the teacher and students is written verbatim. We use the Knowledge Quartet as a framework to investigate what insights about prospective teachers' professional knowledge for mathematics teaching lesson plays can reveal. Our study shows that lesson plays provide thick information about the prospective teachers' ability to transform and connect mathematics in order to make it accessible for students. One lesson play and its analysis are presented in their entirety to give insights into the data and our use of contributory codes from the Knowledge Quartet in the analysis. Implications of the study for teacher education are considered in the conclusion.

Keywords · mathematics teacher education · preservice teachers · mathematical knowledge for teaching · lesson plays · teaching context

Introduction

In this study, we analyse prospective student teachers´ (PSTs') planning documents for field practice. More precisely, we study their lesson plans, written in the form of lesson plays. Zazkis, Liljedahl, and Sinclair (2009) introduced the concept of the lesson play as an imagined discussion between a PST and his or her students, written verbatim. Lesson plays are of interest in teacher education for a number of reasons. John (2006) argued that traditional lesson plans do not give insights into "the substance of the particular activity" (p. 487), such as the definitions and mathematical language planned to be used to describe the concepts and relationships involved, the possible difficulties one can expect students to experience, and students' ideas. Zazkis et al. (2009; Zazkis, Sinclair, & Liljedahl, 2013) claim that lesson plays can give an opportunity for in-depth discussions of crucial aspects of mathematics teaching before the lesson, while such discussions can only take place afterwards when using a traditional lesson plan. As such, writing lesson plays aligns with the position of Morris, Hiebert, and Spitzer (2009) and their work with PSTs unpacking learning goals as a part of mathematical knowledge that is "well suited to pre-teaching experiences" (p. 495).

In this study, the work on lesson plays emphasises setting clear mathematical goals for the planned discussion, selecting tasks and using mathematical questions and representations to ensure that these goals are reached. As such, the lesson plays are used to promote essential elements of productive mathematical discussions (e.g., Smith & Stein, 2011).

Another reason for work with lesson plays is that the writing of imagined discussions does not depend on the knowledge of particular students or what teaching methods they are accustomed to. Thus, such writing can be a good option for coursework in mathematics teacher education where such knowledge is often limited. Working with the different competencies needed to plan for the important practice of leading mathematical discussions can give teacher educators fertile ground for studying and improving both the PSTs' subject matter knowledge and their ability to plan for "ambitious teaching" (Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010).

The purpose of this study is to further develop an understanding of what lesson plays can reveal about PSTs' professional knowledge in terms of mathematics teaching. Crespo, Oslund, and Parks (2011) argued that written imagined classroom discussions give insights into PSTs' development of mathematics teaching practice in a way that is difficult to obtain using other forms of lesson plans. Zazkis et al. (2013) analyse several lesson plays and show that PSTs' understanding of mathematical concepts and ideas appear clearly in their writing. In addition, the PSTs' pedagogical moves (such as the types of questions, planned responses to student reasoning and the inclusion of student ideas) are apparent. In the study by Zazkis et al. (2013), the focus of the lesson plays centred on student misconceptions. In this study, we expand the work of Zazkis et al. (2013) by analysing lesson plays where the purpose is to develop and discuss specific arithmetic strategies. Further, we use the Knowledge Quartet as a basis for the analysis. The Knowledge Quartet (Rowland, Huckstep, & Thwaites, 2005) is a broad framework for studying ways in which the mathematics-related knowledge of prospective teachers comes into play in the classroom. Our research question is: *What insights about PSTs' professional knowledge for mathematics teaching can be gained from analysing their lesson plays?*

The study is conducted at one of the largest university colleges in Norway, with undergraduate students attending a course for prospective teachers preparing to teach grades 1 to 7.

Theoretical Framework

Lee Shulman, one of the pioneers in describing the knowledge needed for teaching, introduced the notion of *subject matter knowledge* and *pedagogical content knowledge* to denote the types of teacher knowledge specific for teaching a particular subject (Shulman, 1986). In this study, we use the term "professional knowledge for mathematics teaching" to denote these two notions for the case of mathematics. Subject matter knowledge is knowledge about concepts, procedures and relations in mathematics, along with knowledge about the nature of mathematics and its development. Pedagogical content knowledge is knowledge regarding how to make mathematical topics accessible to students alongside knowledge about particular challenges in teaching and learning mathematics.

Building on Shulman's work, Rowland, Huckstep, and Thwaites (2005) described how teachers' subject matter knowledge and pedagogical content knowledge surfaced in teaching mathematics. Analysing videos from the teaching of mathematics, they developed a framework called the Knowledge Quartet that "provides a means of reflecting on teaching and teacher knowledge, with a view of developing both" (Turner & Rowland, 2011, p. 197). The Knowledge Quartet comprises four dimensions (foundation, transformation, connection and contingency), each with its own contributing categories, known as codes (see Appendix). For each dimension, the contributory codes describe situations or aspects of teaching where a teacher's mathematical knowledge for teaching surfaces.

The foundation dimension refers to a teacher's theoretical knowledge of mathematics and beliefs about mathematics: what it is, why it is important and what mathematics should be taught in schools. This knowledge is learned at school, in college and during teacher education. A key notion in this theoretical dimension is that knowledge of mathematics and beliefs about mathematics have the "potential to inform pedagogical choices and strategies in a fundamental way" (Turner & Rowland, 2011, p. 200). Theoretical knowledge can surface in, for example, the PST's use of mathematical language and in how the PST concentrates on developing understanding rather than focusing on procedures. It is important to note that beliefs about mathematics and mathematics education are included in the dimension as they play a significant role in teaching.

The transformation dimension goes to the heart of teaching a subject. It relates to how a teacher "re-presents" and unpacks content through using examples, illustrations, demonstrations, activities and questioning. Turner and Rowland refer to the writing of Ball (1988), "who distinguishes between knowing some mathematics 'for yourself' and knowing in order to be able to help someone else learn it" (Turner & Rowland, 2011, p. 201). An essential part of this dimension in our analysis is to what extent the PST chooses appropriate examples and representations to help students to learn a mathematical idea. In addition, central to this dimension is the way in which PSTs plan to use questions to elicit and deepen students' understanding of a concept or procedure.

The next dimension, connection, is about situations in educational work that relate to connections within both mathematical content and teaching. It is about connections across teaching episodes, between lessons or lesson series. It binds together choices and decisions taken about separate parts of the mathematical content, for example, by stating relationships between various procedures and strategies. The mathematical knowledge relevant for this dimension can surface in the PSTs' planning of how to sequence tasks and examples, and in links between tasks.

The final dimension, called contingency, is about a teacher´s preparedness and ability to respond to unexpected events in a thoughtful and justifiable way. This dimension is not relevant to our study, since contingency is about unplanned events that appear only in actual teaching.

The Knowledge Quartet framework was developed by analysing videos of teaching and is mostly used in research to analyse episodes (often videos) of authentic teaching (e.g., Rowland et al., 2005; Petrou, 2008; Turner, 2012; Rowland & Zazkis 2013; Kleve & Solem, 2014). We, however, in line with Huntley (2011), use the Knowledge Quartet to analyse teacher knowledge based on the planning of teaching and not on actual teaching. Huntley (2011) analysed both planning documents and teacher interviews, while in our study, the analysis is only based on planning documents.

Method

Participants and the course

The PSTs participating in this study were in their first year of a 4-year undergraduate study programme for prospective teachers preparing for teaching grades 1 to 7. A large majority of them had just one year of mathematics from upper secondary school and no other additional courses in mathematics before entering teacher education. The study was conducted within the frames of a compulsory course in mathematic didactics. The course comprised approximately 18 4-hour sessions. The main topic was multiplicative thinking, with an emphasis on strategies, properties and reasoning in multiplication and division. Alongside this, a great deal of time was devoted to discussions about factors that might promote productive mathematical discussions with students in the classroom. The use of representations in the form of illustrations and contexts as tools for reasoning was stressed. During the course, the PSTs worked with four assignments that involved lesson plays. The work extended from analysing and discussing ready-made lesson plays in class to individually writing their own lesson plays. The first three lesson plays concerned strategies for and properties of multiplication. For the fourth play, the PSTs could choose to write a lesson play focusing on either multiplication or division. This assignment was given near the end of the course and the data for this study are drawn from it. The PSTs acted out all four lesson plays during short school placements.

Design of the assignment

In working with PSTs, it is a challenge to develop assignments for school placements that are manageable, without reducing the complexity of either the mathematics or of the teaching. The assignments should make it possible to explore the essence of important topics in mathematics and mathematics teaching. In the school-placement assignment we analyse herein, the PSTs were asked to write a lesson play for a 10 to 20 minute-long discussion addressing a strategy or property of multiplication or division based on a string of arithmetic problems. Leading mathematical discussions based on such strings is one of the activities proposed by Lampert et al. (2010) as "packages" of *core practices* (Grossmann, Hammerness, & McDonald, 2009). Work on arithmetic strategies was a substantial part of the university-based coursework and the assignment can be seen as a form of approximation to practice (Grossmann et al., 2009). The PSTs were explicitly asked to define a mathematical goal for the discussion and choose a suitable string of arithmetic problems. Next, the PSTs were asked to write an "ideal productive discussion" verbatim including their vision of what they would write on the blackboard. The discussion was supposed to be "ideal" with regard to achieving the mathematical goal, not one in which students would always give the correct answer. The term "productive discussion" is used as in Chapin, O'Connor and Anderson (2009) to describe mathematical discussions that provide opportunities for students to reason and develop conceptual understanding. They suggest the use of *Talk Moves* to help make discussions productive. Examples of Talk Moves are repetition and revoicing student utterances, asking for elaboration, reasoning and revision, giving waiting time and using pair-talk. In the coursework, the notion of productive discussions and the use of Talk Moves were discussed explicitly through the analysis of several classroom episodes.

In comparison with the lesson plays proposed by Zazkis et al. (2009), our assignment put more emphasis on defining a clear mathematical goal and shaping the discussion so that it would meet that goal.

Data

A total of 147 individually written lesson plays were handed in for the assignment. The large majority of these addressed a property or strategy of multiplication. To a large extent, we found that these lesson plays imitated lesson plays that had already been discussed during the coursework, making them less suitable for analysing the PSTs' professional knowledge for teaching. Therefore, we chose to limit our data to the 23 lesson plays concerning division, as these can be seen to more genuinely represent the PSTs' knowledge.

Data analysis

The data analysis was conducted in three steps. First, the four researchers individually analysed three randomly chosen lesson plays using codes (see the Appendix) from the Knowledge Quartet. We then met to discuss our analysis. The purpose of this was to become familiar with the data and the analysis framework, and to develop an agreed interpretation of different aspects of the lesson plays using notions from the Knowledge Quartet framework. In the second step, we reverted to analysing the 23 lesson plays independently, before again comparing and discussing our analyses. In this step, we discussed which codes were not detectable in our data material, and for those that were, we found it necessary to develop categories to better compare and summarise our analysis. For instance, for the code "theoretical underpinning of pedagogy", the possible categories were "elements of" and "weak", and for the code "choice of examples", the possible categories were "appropriate for applying and discussing the strategy" and "less appropriate". In the third step, we individually re-analysed all 23 lesson plays using the categories and met to compare and discuss our findings. In cases of disagreement, we discussed the given lesson play and agreed on a coding.

Using the Knowledge Quartet in Lesson Play Analysis

We present one lesson play in its entirety to give an idea about the data and our analysis of it. This lesson play, written by Hilde[1], was chosen because it is typical of the sample and suitable for a discussion of our findings. Hilde´s writing in Norwegian is inaccurate in terms of vocabulary, grammar and sentence structure. We have tried to preserve this in the translation.

[1] All PSTs and students in the paper are given pseudonyms.

Some of the imprecise wording is written as student utterances and can be considered reasonable to expect in a discussion with students.

Hilde defines the mathematical goal for the discussion to be to "Increase/decrease one of the numbers in division. To discuss various maths problems, the goal is that students are able to see connections between the position of the number and how to make the arithmetic problem easier by increasing/decreasing the numbers". She plans to use the tasks 15 : 1.5, 15 : 3, 15 : 6, 24 : 4 and 24 : 0.4[2]. Her imagined ideal discussion goes as follows:

Teacher writes on the blackboard 15 : 1.5.

(1)Teacher: When you have figured this out you can put a finger on your nose.

Wait 25 seconds.

(2)Teacher: Klara, what are your suggestions?

(3)Klara: Hmm, I think the answer will be 10, because 1.5 x 10 is 15.

(4)Teacher: Is there anyone else who found a different answer?

(5)Students: (Nobody says anything)

(6)Truls: I agree with Klara when she said that 1.5 x 10 is 15.

(7)Teacher: Good! Because Klara is absolutely right.

Teacher writes on the blackboard 15 : 3 just below the first problem.

(8)Teacher: Now you can try to figure this out, and afterwards you can put your finger on your nose again. (Waiting 10 seconds until everyone has figured out the answer.)

(9)Tine: The answer will be 5.

(10)Teacher: How do you know?

(11)Tine: Yes because 5 x 3 is 15.

(12)Teacher: That is right, but can someone give me another way to figure it out?

(13)Gøril: 3 +3 +3 +3 +3 is 15. I took step by step and calculated how many 3s that fit inside 15.

(14)Teacher: Good! But if we look at the previous calculation, how can we then easily find the answer?

Students are thinking for about 20 seconds.

(15)Klara: I know! 1.5 is half of 3. Before the answer was 10, now that we have double the number we divide the answer must also be half as small.

(16)Teacher: Great idea Klara! Would you Truls retell what Klara came up with?

(17)Truls: I do not know if I understood it correctly, but there was something about that half of the previous, the answer had to be less because we divided by more. Did I understand it correctly?

(18)Teacher: Mmm. We see on the blackboard here that 1.5 is half of 3. Now we divide by more than earlier. We can now see that since we divide by exactly twice as much, the answer is halved. Because we share what we have with more people, then each gets less. Did everyone understand?

Students: Almost all nod and seem to have understood at least a bit.

Teacher writes on the blackboard 15 : 6.

[2] In Norway, the colon denotes division.

(19)Teacher: If you now think for a while and use the same strategy as Klara just used. How can you then easily calculate this task?

Students think for 20–30 seconds.

(20)Gøril: Oh, now I think I understand. 6 is twice as large as 3, so the answer is half as small as we divide by more. Before, the answer to 15 : 3 = 5. 15 : 6 then must be 2.5 because it is twice as big!

(21)Teacher: Very good! Mari, can you draw a story on the blackboard that explains what Gøril just said?

(22)Mari: Well, first we had 15 oranges that were supposed to be divided amongst 3 people, they got 5 oranges each. But then another 3 friends came along who also wanted oranges, then it became a little more difficult to share. They knew from before that 3 people had 5 oranges each. In order to share these, each person who already had oranges had to give first 2 oranges and then cut the last in a half. When all of the 3 first people had done this for the new people who came, then all had been given 2.5 each.

(23)Teacher: Good! This was a long nice story.

Teacher writes 24 : 0.4 on the blackboard.

(24)Teacher: This is maybe a difficult problem to work out, but really, it does not need to be so terribly difficult. Is there any way to make the number larger or smaller so that it becomes easier to calculate?

Students think for 20 seconds.

(25)Mari: I am not sure if I am thinking correctly, but if we take the 24 : 4, it is certainly easier to calculate. But I am not sure what to do with the zero and the comma.

(26)Teacher: You are on to something very reasonable here, Mari, do you want to write it on the blackboard? And can someone help her continue?

(27)Truls: I know what you can do, you just put the zero behind the answer. That is, you think that you calculate 24 : 4 and that is 6, but you actually calculate 24 : 0.4. The zero you took away, you just put it right behind the number 6, so that the answer becomes 60.

(28)Teacher: Yes, but why can you just put it behind?

(29)Truls: I do not know, that is just the way it is.

(30)Gøril: I think I understand why. 4 is 10 times as big as 0.4. So when we divide by such a small number as 0.4, the number becomes 10 times bigger than if we just calculated 24 : 4.

(31)Teacher: Good Gøril! It is easier to calculate in your head 24 : 4, and then we get the number 6. Since we know that 4 is 10 times as big as the 0.4, we know that 6 has to be multiplied by 10, in order to find the correct answer. So when we calculate 24 : 0.4, we can imagine that we first calculate 24 : 4 = 6 and then we take 6 x 10 = 60 that will be the answer we get.

(32)Teacher: Any questions?

The students shake their heads.

We will now use the Knowledge Quartet as a lens for analysing Hilde's lesson play. We use italics to indicate the contributory codes for each dimension.

Foundation

This lesson play focuses on a specific strategy for division: scaling the divisor up or down. At the heart of this strategy lies the inverse proportionality between the divisor and quotient when the dividend is fixed (e.g., speed and journey time for a given journey). Although rarely stated as a general strategy, it underpins the usual algorithm for division by a non-integer finite decimal number, as Truls and Gøril indicate (turns 27 and 30) and the teacher explains (turn 31). The objective of the discussion is not very clearly stated, but since a story is used to reason about the numerical relationships (e.g. turn 22), we assume that the purpose is to discuss and justify why the strategy works and is not only about practising the strategy. If so, then Hilde shows an *awareness of purpose* by introducing the story and by emphasising that it is not enough just to get an answer or give instructions about how to use the strategy (e.g. turns 10 and 28). Hilde seems to expect the students to take an active part in explaining and defending their strategies (e.g. turns 10 and 12) and she presumes that the students will listen to each other's reasoning (e.g. turn 16). Hence, the lesson play indicates that Hilde has ideas about how mathematics is taught and learned and is *theoretically underpinned*. Understanding the nature of mathematics (and of mathematical argumentation in particular), the use of different representations, and the importance of connections between representations is also part of the theoretical underpinning of pedagogy. The targeted strategy in this lesson play is "discovered" by the students (turn 15) and the teacher helps them to see why it is reasonable by introducing a story (e.g. turn 22). This suggests that Hilde attaches importance to building conceptual understanding and that she believes that stories are useful representations for this purpose. However, the reasoning in the lesson play consists mainly of descriptions of what to do with the numbers and statements about some relations between numbers, without focusing on the operation involved. For example, in the summary in turn 31, the teacher says, "Since we know that 4 is 10 times as big as the 0.4, we know that 6 has to be multiplied by 10, in order to find the correct answer." Consequently, to some extent, the lesson play *concentrates on procedures.*

The *mathematical terminology* used in the lesson play is rather imprecise. For example, in turn 24, the teacher talks about "making the number larger or smaller", when she means scaling the divisor. Hilde uses the words "double", "halves" and "ten times bigger" correctly, but there is no attempt to explicitly discuss the meaning of the mathematical terms. However, the teacher summarises what students have said several times and expands their explanations using a more formal mathematical language (see e.g. turns 15 to 18 and 30 to 31). When defining the mathematical goal, the language is particularly imprecise. Hilde uses "one of the numbers" instead of "the divisor", and it is unclear what "the position of the number" refers to.

Transformation

Hilde uses three pairs of arithmetic problems in the discussion. The first one considers the relation between $15 : 1.5$ and $15 : 3$, the second between $15 : 3$ and $15 : 6$ and the third between $24 : 0.4$ and $24 : 4$. The examples appear to have somewhat different purposes, and we discuss the *choice of* each *example* in relation to the purpose (as we have identified it) of the lesson play. The first example ($15 : 1.5$ and $15 : 3$) is used to direct students' attention to the targeted strategy (turn 14). The first arithmetic problem involves a decimal number; hence, it is more complex than the second one is. Using $15 : 1.5$ to calculate $15 : 3$ is unnatural and unnecessary (turn 11), but if the purpose is to emphasise the relation between divisors and quotients with a fixed dividend, it is reasonable to start with two problems one can easily calculate without using the strategy in order to compare and contrast. If so, the arithmetic problems could be chosen so that the relation is even more obvious, for example, by using $24 : 3$ and $24 : 6$. The main purpose of the second example seems to be about the reasoning behind the strategy. The problem $15 : 6$ is not so easily solved by number facts; hence, it is a good idea to build on the previous task, even though there may be other strategies that are equally efficient. The teacher explicitly asks the students to use the targeted strategy in turn 19 and then directs the discussion towards reasoning by asking for a story (turn 21).

The third example involves scaling by a factor of 10. Since the previous discussion was about scaling by a factor of 2, Hilde's intention here might be to point out that it does not matter by what factor you increase or decrease the divisor, the answer will decrease or increase by the same factor. However, this is not emphasised in the lesson play, and we discuss this further in "Connection". Moreover, if the aim is to discuss scaling the divisor as a general strategy, it is restricting to only use the factors 2 and 10 in the examples since both 2 and 10 are somewhat special, the first indicating doubling and the second using the position system. Another purpose of the third example might be to illustrate the usefulness of the strategy. For this purpose, the example is suitable since it reduces complexity in the calculation and at the same time provides an opportunity to discuss the often-used "rule" about moving the decimal point in similar tasks.

The *use of* stories as a *representation* also needs to be seen in relation to the mathematical goal of the discussion. Both stories (turns 18 and 22) use the partition model of division. It is not clear whether this is a conscious choice or a result of Hilde's limited knowledge about possible models in division. For the second example (15 : 3 and 15 : 6), it is an appropriate model for reasoning, but for the other two examples, this model is more difficult to use, since the divisors are not natural numbers. A story problem based on the quotative model would work for all three examples and utilising the same story problem could have made a stronger connection between them. The intention of the story introduced in turn 22 is to help students reason about why the strategy works for 15 : 3 and 15 : 6. However, the connection between the story and the strategy is weak in the sense that the story talks about "three more people" and "2 oranges and half an orange", and it is not explicitly stated that the number of people has doubled from 3 to 6 and that the number of oranges per person is halved from 5 to 2.5. It is a student who expands the story connecting the numbers to a real-life situation, but the teacher does not clarify the student´s utterances.

Even though the lesson play focuses on procedures to some extent, we find no examples of explicit *teacher demonstrations.* The students are involved in both inventing the strategy and in reasoning for the strategy.

Connection

The strategy that Hilde brings up in her lesson play can be challenging for students to grasp, and the goal of the discussion is to understand the rationale behind this strategy, not just to use it. Understanding demands connecting concepts and using and connecting different representations. The discussion Hilde outlines moves forward quite smoothly and quickly, yet the students do not get time to talk and investigate the relations in pairs or groups, or to use paper and pen to help their thinking. This may indicate that Hilde does not *anticipate* enough *complexity* in the discussion. To some extent, she presumes that understanding the relationship between the divisor and quotient will be challenging, and she mitigates this obstacle by bringing a story into the discussion. However, the story is actively used only for one of the examples. Likewise, it appears that Hilde is not aware of the conceptual complexity involved in generalising the strategy from using one factor (2 in this case)to using some other factor, here 10. In turn 24, the teacher makes a connection between these two by using the notion of smaller and larger numbers, but the similarities and differences are not discussed explicitly, and the language used in the two cases is different. For the scaling factor of 2, words such as "twice," "double," and "half" are used, while "10 times as big as" and "multiplied by 10" are used when the scaling factor is 10.

Hilde organises the discussion based on the three examples, where the first is used mainly to focus students' attention on the targeted relation/strategy, the second to justify the strategy and the third to generalise and emphasise the usefulness of the strategy. The *decision about sequencing* "attention – reasoning – use" seems suitable in discussions on calculation strategies, even though the examples themselves are somewhat problematic.

There are opportunities for *making connections between procedures* and *between concepts* in the lesson play, for instance, by connecting multiplication and division (turns 3 and 11) and connecting addition and multiplication (turn 13), but the connections are not emphasised or made explicit. The strategy proposed in turn 13 is repeated addition, which links to quotative division, but the stories given (turns 18 and 22) indicate partitive division. As repeated addition is not the focus of the conversation, it is reasonable that this strategy is not prioritised. However, as discussed above, a partitive model of division would be more appropriate for reasoning about the examples chosen and connecting repeated addition with the targeted strategy could be a way of organising the discussion further.

Findings

Our research question asks what insights about professional knowledge for mathematics teaching can be gained from analysing PSTs' lesson plays? Since our analysis of the 23 lesson plays is based on the Knowledge Quartet framework, we organised our findings according to the dimensions of the framework. For each dimension, we present a table showing how we have categorised the lesson plays with respect to the contributory codes, before we elaborate on the findings related to each code. We point out that the aim of the tables is neither to develop categories for the codes, nor to give a quantitative overview of the quality of the lesson plays. The tables are merely used as a tool to enable us to more clearly identify what aspects of teacher knowledge can be accessed in lesson plays.

Foundation

Table 1 lists the codes of the foundation dimension of the Knowledge Quartet relevant to this study, and the number of PSTs whose lesson plays fell into each code.
Of the seven codes constituting the foundation dimension of the Knowledge Quartet, the codes adherence to textbook and identifying errors were not applicable in our study due to design of the assignment. We found no examples of overtly displayed subject knowledge in our material. This might partly be due to the design of the lesson plays. It is hard to see what this knowledge could look like when discussing a particular calculation strategy, without unexpected contributions or questions from students. A knowledgeable teacher, however, could potentially seize an opportunity to display such knowledge. All but one of the lesson plays stated

"reasoning" as part of their goal for the discussion. Most of the lesson plays were thus considered to show an awareness of purpose, based on their attempt to bring in a story to reason for the strategy in focus. In three cases, we saw a lack of awareness, resulting in dialogues that entirely focussed on procedures, even though the stated goal was to focus on conceptual understanding. Most of the lesson plays drew to some extent on knowledge of well-established results in mathematics education research regarding the way in which pupils learn mathematics. These were therefore considered to show elements of theoretical *underpinning of pedagogy*. This was identified by the way the PSTs imagined that the students took an active role in reasoning through the use of repeating and revoicing, by how the teacher asked for justification a number of times and by the fact that several representations were introduced during the discussion. This could be expected, as the notion of productive mathematical discussions, the use of talk moves to support reasoning and the use of different representations to promote understanding were all part of the theory the students were exposed to during coursework. Nevertheless, five of the lesson plays showed a weak theoretical underpinning of pedagogy. What distinguished these from the rest was that they gave their students very limited opportunities to think and reason, and they made no attempt to connect the targeted strategy to a representation.

Table 1
Relevant codes of the foundation dimension of the Knowledge Quartet with the number of PSTS whose lesson plays are in each category.

Awareness of purpose	Awareness 20	Lack of awareness 3
Concentration on procedures	No 7	To some degree 16
Overt display of subject knowledge	Yes 0	No 23
Theoretical underpinning of pedagogy	Elements of 18	Weak 5
Use of mathematical terminology	Mostly precise 6	Imprecise 17

For the 18 lesson plays that did show elements of theoretical underpinning, there were, of course, differences in how this was played out. A majority of them did not use the Talk Moves very efficiently to expand student thinking, and the use of representations and the argumentation provided were not considered strong enough to promote conceptual understanding. Thus, a *concentration on procedures* was emphasised more in the dialogue than the stated goal for the discussion indicates.

Since the mathematical goal for the lesson was stated explicitly and the planned discussion was written verbatim, the lesson plays provided very good insights into the PSTs' *use of mathematical terminology*. In the majority of the lesson plays, the language was partly imprecise, but not incorrect. It is possible that the PSTs sometimes used imprecise terminology intentionally, as an approach to the students' terminology, but in several lesson plays, the lack of any attempt to clarify students' utterances made the discussion diffuse. In the wording of the mathematical goal, the terminology, more often than not, was imprecise. There should be no reason to formulate the goal of the lesson in a student-friendly language, and we see this as a shortfall in the PSTs' knowledge of the correct mathematical terms and their precise meanings.

Transformation

Table 2 lists the codes of the transformation dimension of the Knowledge Quartet relevant to this study, and the number of PSTs whose lesson plays fell into each code.

Table 2
Relevant Codes of the transformation dimension of the Knowledge Quartet with the number of PSTS whose lesson plays are in each category.

Choice of examples	Appropriate for applying and discussing the strategy	Less appropriate
	17	6
Choice of representation	Good connection between representation and strategy	Weak connection
	4	19
Teacher demonstration	No	Yes
	20	3

Of the four codes constituting the transformation dimension of the Knowledge Quartet, the *use of instructional material* was not applicable in our study. The code *choice of examples* was very prominent in the PSTs' lesson plays. The majority of PSTs chose a string of examples that was appropriate for exploring the targeted strategy. In most cases, we could guess the strategy addressed simply by observing the string. This was the case in Hilde's lesson play focusing on scaling the divisor. Another example was a lesson play where the string "100 : 4, 36 : 4, 136 : 4, 120 : 12, 48 : 12, 168 : 12" shows that splitting the dividend in two parts was to be considered. As discussed in the case of Hilde, even though the chosen string of examples was suitable for exploring the targeted strategy, the target strategy might not always be the most efficient, and the examples might be problematic, for instance, with respect to generalising the strategy. This does not show in Table 2.

The lesson plays were also very informative when it came to *choice* and the use *of representations.* In addition to mathematical symbols, most of the PSTs used stories, sometimes supported by an illustration, to explain and argue for the targeted strategy. The appropriateness and the actual use of the representations varied considerably. Some would have benefitted from choosing a different model for division, as discussed in the case of Hilde, but most PSTs chose a model that could work for the chosen string of examples. However, very few of them managed to connect the story and the strategy in a way that would build conceptual understanding. The majority of the lesson plays were therefore considered weak when it came to connecting representation and strategy. Hilde's lesson play was typical in this sense. In her story, the important relationships between the numbers became too diffuse. Only four of the lesson plays made the connections more explicit. One example is the story "If 4 siblings get 100 kroner[3] to divide equally between them one week, and then 36 kroner the next week, they would have split 136 kroner equally between them altogether." This showed very clearly why the results of the two division tasks 100 : 4 and 36 : 4 can be added to get the result of the division task 136 : 4.

Since the imagined discussion was written verbatim, the extent to which the *teacher* used *demonstration* was evident. Even though 16 of the lesson plays focussed on procedure to some extent, only three were considered to have examples of clear teacher demonstrations, where the teacher was doing most of the exploring and reasoning. This can, to some extent, be explained by the requirement of the discussion to be productive.

Connection

Table 3 lists the codes of the connection dimension of the Knowledge Quartet relevant to this study, and the number of PSTs whose lesson plays fell into each code.

Table 3
Relevant Codes of the connection dimension of the Knowledge Quartet with the number of PSTS whose lesson plays are in each category.

Anticipation of complexity	To some extent 11	No 12
Decisions about sequencing	Appropriate 23	Not appropriate 0
Making connections between procedures	Explicit 1	Not explicit 22
Making connections between concepts	Explicit 0	Not explicit 23

The lesson plays provided a good insight into the PSTs' *anticipation of complexity.* We found that approximately half of the PSTs rarely anticipated any major conceptual challenges of significance for the students at all. Occasionally the students gave the wrong answers, but these were just computational errors, easily amended by the teacher or other students. The discussion proceeded smoothly, and time was spent on the more trivial examples rather than on scrutinising the core of the strategy. Lesson plays that had the generality of a strategy as the stated goal of the discussion typically addressed this by the teacher asking, "Does this work for all numbers?" with the answer "Yes" from all students. This indicated a lack of awareness of the complexity of stating a general claim. To some extent, the other half of the lesson plays

[3] Kroner is the Norwegian currency.

anticipated the complexity by using more time for investigation and argumentation and less time for the more trivial examples.

The *decisions about sequencing* that PSTs make were evident in their lesson plays, but we found little variation regarding this issue. This may be due to the design of the assignment. Once the problems were chosen, the sequencing was rather straightforward, and all PSTs in our study chose sequencing that was reasonable. The lack of variation in the data may indicate that our insights about this element of mathematical knowledge was impeded, since sequencing was almost given by the design.

We found it hard to evaluate the PSTs' *recognition of conceptual appropriateness* based on the lesson plays. We, and the PSTs, had limited information about the group of students where the lesson play was going to be enacted. We have therefore chosen not to evaluate the lesson plays with respect to this code. We found, however, that the string of examples the PSTs chose varied with respect to the cognitive demands they put on the students. A string such as "12 : 3, 120 : 30, 1200 : 300" might seem too trivial to a class of 7th-grade students, while it seems unreasonable to expect that 6th-graders would be able to utilise the property discovered by looking at the relation between the two tasks 30 : 6 and 60 : 12 and develop a strategy for solving the task 9 : 1.5.

In the lesson plays, it was clear which *connections between procedures/concepts* were emphasised and how these connections were made. In the majority of the lesson plays, there were attempts to make connections between a model for division and a strategy for division. We chose not to count these as "making connections", as this aspect was already handled under "use of representations". As was the case in Hilde's lesson play, there were opportunities to make connections both between procedures and between concepts in most of the lesson plays, but these connections were not made explicit. The only exception was a lesson play discussing the strategy of splitting the dividend. Here, the teacher explicitly asked the students whether one could also split the divisor (i.e. if one could solve 240 : 4 by adding the results of 240 : 2 and 240 : 2). One student stated that in multiplication, one can split either of the factors (the distributive law). The students reasoned about the similar claim for division when using a money context and concluded that one cannot split the dividend. We counted this as an explicit example of making connections between procedures (contrasting strategies of multiplication and division).

Discussion and Implications

We found that the lesson plays analysed in this study gave good insights into several aspects of PSTs' professional knowledge for mathematic teaching, especially related to the transformation and the connection dimensions of the Knowledge Quartet. The design of the assignment, where the PSTs had to state the mathematical goal of the lesson explicitly and write the intended interaction between the teacher and students verbatim, made it possible to analyse the (planned) lesson in depth. The lesson plays gave insights into the PSTs' ability to state a reasonable mathematical goal for a short discussion, and if their choice of examples and representations had potential to promote the goal. Further, the lesson plays provided insights into whether or not the PSTs were able to utilise the potential of the examples and the representations they had chosen. Smith and Stein (2011) (see also Kazemi & Hintz, 2014) emphasised the importance of teachers clarifying the mathematical goal for themselves before planning a discussion. The mathematical goal is crucial in making decisions about what tasks to choose, what representations to use, what questions to ask, and what to highlight in the discussion.

As the dialogue was written as an ideal discussion, the lesson plays revealed the PSTs' ability to use mathematically correct language and what connections they consider important to highlight. It also revealed the PSTs' anticipation of complexity through how they structure the dialogue and what questions they address.

The requirements put on the content and the form of this particular lesson play also entailed some limitations regarding some aspects of teacher knowledge. The lesson play should have a clear mathematical goal and it should be formed as a productive discussion based on a string of related arithmetic problems. Thus, it is not possible to say anything about how PSTs would have taught if they had been free to choose the content and method. A less structured assignment could possibly have revealed more about the PSTs' beliefs about, and understanding of, what mathematics is and how it is best learned.

By analysing these lesson plays, we could point to deficiencies in PSTs' professional knowledge regarding mathematics teaching. However, we could not infer the reasons why the PSTs chose to include particular parts of their dialogue based on these data. This is a limitation of our study. Despite this, we find that lesson plays can be useful tools for teacher-educators' understanding of PSTs' knowledge regarding both mathematics and the teaching of mathematics. Our data revealed some critical aspects of PSTs' development of mathematical knowledge for teaching. An example of this is the *choice (and use) of examples and representations.* Providing clear explanations is a difficult part of teaching mathematics, as different representations need to be coordinated, the explanation has to be correct and students' understanding and responses have to be taken into account (Kinach, 2002; Stylianou, 2010; Charalambous, Hill, & Ball, 2011). Another critical aspect that we found in our data was the *anticipation of complexity*. It was challenging for the PSTs to assume the perspective of a student discussing a mathematical idea. This has implications for future work with PSTs. They need more practice at providing explanations, connecting representations and unpacking complexity. Further, teacher educators need to design activities which address these aspects. Lesson plays do not give information about the contingency dimension of the Knowledge Quartet. However, by giving PSTs several opportunities to write and act out lesson plays, we believe they will be better able to handle unexpected situations. Experiences from the first acting out can reveal the need to do more in-depth lesson planning. It is expected that the PSTs will gradually experience fewer unpredicted situations and tackle the ones that do appear better.

The critical aspects we identified in our analysed lesson plays coincided with aspects identified in studies about the development of professional knowledge for mathematics teaching based on analyses of teaching episodes, as in Rowland et al. (2005) and Nicol (1999). We note that these critical aspects emerge in planning instruction through writing lesson plays. In this way, teacher educators gain an opportunity to address such aspects before PSTs actually teach, giving teacher educators fertile ground for improving PSTs' mathematical knowledge for teaching.

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Appendix: The Knowledge Quartet – dimensions and contributory codes (Rowland, 2014)

Dimension	Contributory codes
Foundation	Awareness of purpose
	Adheres to textbook
	Concentration on procedures
	Identifying errors
	Overt display of subject knowledge
	Theoretical underpinning of pedagogy
	Use of mathematical terminology
Transformation	Choice of examples
	Choice of representation
	Use of instructional materials
	Teacher demonstration (to explain a procedure)
Connection	Anticipation of complexity
	Decisions about sequencing
	Recognition of conceptual appropriateness
	Making connections between procedures
	Making connections between concepts
Contingency	Deviation from agenda
	Responding to students' ideas
	Teacher insights during instruction
	Responding to the (un)availability of tools and resources

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